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CONTENTS

Relative extrema of Legendre functions of the second kind	<i>Richard Askey</i>	1
Multipliers for the Weyl transform and Laguerre expansions	<i>S Thangavelu</i>	9
On the inverse Laplace transform of the product of a general class of polynomials and the multivariable H-function	<i>K C Gupta and R C Soni</i>	21
Compact operators on a Banach space into the space of almost periodic functions	<i>M V Deshpande and M N Parekh</i>	25
Verschiebung and Frobenius operators	<i>K Varadarajan</i>	37
Criterion for smoothness of Schubert varieties in $Sl(n)/B$	<i>V Lakshmibai and B Sandhya</i>	45
Vector fields and framings on isolated complete intersection singularities	<i>José Seade</i>	53
A class of totally geodesic foliations of Lie groups	<i>G Santhanam</i>	57
On infinite-dimensional control systems with state and control constraints	<i>Nikolaos S Papageorgiou</i>	65
$\tilde{K}'_3$ — A new triangulation of R^n	<i>D P Shukla and N P S Bawa</i>	79
On shock dynamics	<i>Phoolan Prasad</i>	87
Corrections to some expressions in "On the propagation of a multi-dimensional shock of arbitrary strength"	<i>R Srinivasan and Phoolan Prasad</i>	93
Singular pencils of quadrics and compactified Jacobians of curves	<i>Usha N Bhosle</i>	95
Decomposition of the de Rham complex	<i>V Srinivas</i>	103
On the ratio of two blocks of consecutive integers	<i>N Saradha and T N Shorey</i>	107

Extension of certain types of generating relations	<i>R K Raina</i>	133
Some uncertainty inequalities	<i>S Thangavelu</i>	137
Riesz means for the sublaplacian on the Heisenberg group ..	<i>S Thangavelu</i>	147
Pairs of II_1 factors	<i>V S Sunder</i>	157
Existence of solution for nonlinear Volterra integral equations.....	<i>K Balachandran and S Ilamaran</i>	179
Effect of a viscous fluid flow past a spherical gas bubble on the growth of its radius	<i>B B Chakraborty</i>	185
Affine Poisson structures.....	<i>K H Bhaskara</i>	189
On the existence of a finite invariant measure	<i>M G Nadkarni</i>	203
Diophantine approximation by linear forms on manifolds.....	<i>M M Dodson, B P Rynne and J A G Vickers</i>	221
Lebesgue-Stieltjes integral inequalities in several variables with retardation	<i>Xureong Mao</i>	231
Integrability and L^1 -convergence of sine series with generalized quasi-convex coefficients	<i>Suresh Kumari and Babu Ram</i>	245
Some results involving Bessel function and Fox's H-function of two variables	<i>S D Bajpai</i>	255
Köthe spaces and topological algebra with bases	<i>Subhash J Bhatt and G M Deheri</i>	259
Critical Sobolev exponent problem in $\mathbb{R}^n (n \geq 4)$ with Neumann boundary condition	<i>Adimurthi and S L Yadava</i>	275
Anderson model with decaying randomness existence of extended states	<i>M Krishna</i>	285
An efficient algorithm for linear programming	<i>V Ch Venkaiah</i>	295
Erratum		303

Relative extrema of Legendre functions of the second kind

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Abstract. Let $\mu_{k,n}$ denote the relative maxima of $|Q_n(x)|$, the Legendre function of the second kind, ordered so that $\mu_{k+1,n}$ occurs to the left of $\mu_{k,n}$. They by analogy with a theorem of Szegő for Legendre polynomials, $\mu_{k,n+1} < \mu_{k,n}$, $k = 1, \dots, n$, $n = 1, 2, \dots$.

Keywords. Inequalities; monotonicity; Legendre functions of the second kind.

1. Introduction

In addition to the Legendre polynomials $P_n(x)$, the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (1)$$

has a second solution that occurs in some applications. This solution is called $Q_n(x)$ and can be given by many formulas. For x outside $-1 \leq x \leq 1$, one formula is

$$Q_n(x) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{x-t} dt. \quad (2)$$

See [3] and [5] for the formulas used below. Formula (2) can be rewritten as

$$\begin{aligned} Q_n(x) &= \frac{1}{2} \int_{-1}^1 \frac{P_n(t) - P_n(x)}{x-t} dt + \frac{P_n(x)}{2} \int_{-1}^1 \frac{dt}{x-t} \\ &= P_n(x)Q_0(x) - W_{n-1}(x) \end{aligned} \quad (3)$$

where $W_{n-1}(x)$ is a polynomial of degree $n-1$ given by

$$W_{n-1}(x) = \frac{1}{2} \int_{-1}^1 \frac{P_n(x) - P_n(t)}{x-t} dt. \quad (4)$$

For complex z off $-1 \leq z \leq 1$,

$$Q_0(z) = \frac{1}{2} \log \frac{z+1}{z-1} \quad (5)$$

with the argument determined by $Q_0(z) > 0$ when $z > 1$.

For $-1 < x < 1$, $Q_n(x)$ is defined as the average of $Q(z)$ when z approaches x from

$$Q_n(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{Q_n(x + i\varepsilon) + Q_n(x - i\varepsilon)}{2}. \quad (6)$$

This gives

$$Q_n(x) = P_n(x)Q_0(x) - W_{n-1}(x), \quad -1 < x < 1 \quad (7)$$

with

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}. \quad (8)$$

From (2) it is easy to find the three term recurrence relation satisfied by $Q_n(x)$. Recall that $P_n(x)$ satisfies

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 0, 1, \dots \quad (9)$$

Then

$$\begin{aligned} (2n+1)xQ_n(x) &= (n+1)Q_{n+1}(x) - nQ_{n-1}(x) \\ &= \frac{(2n+1)}{2} \int_{-1}^1 \frac{(x-t+t)P_n(t)}{x-t} dt - \frac{1}{2} \int_{-1}^1 \frac{(n+1)P_{n+1}(t) + nP_{n-1}(t)}{x-t} dt \\ &= \frac{(2n+1)}{2} \int_{-1}^1 P_n(t) dt + \frac{1}{2} \int_{-1}^1 \frac{(2n+1)tP_n(t) - (n+1)P_{n+1}(t) - nP_{n-1}(t)}{x-t} dt \\ &= 0 \text{ when } n \geq 1, \\ &= 1 \text{ when } n = 0, \end{aligned}$$

so

$$(2n+1)xQ_n(x) = (n+1)Q_{n+1}(x) + nQ_{n-1}(x), \quad n = 1, 2, \dots \quad (10)$$

Since $Q_n(x)$ as a function of x satisfies the same differential equation as $P_n(x)$, and as a function of n the same difference equation, one can ask if $Q_n(x)$ also satisfies the mixed differential-difference relations $P_n(x)$ satisfies. Here are two such

$$(1+x)[P'_{n+1}(x) - P'_n(x)] = (n+1)[P_{n+1}(x) + P_n(x)] \quad (11)$$

$$(1-x)[P'_{n+1}(x) + P'_n(x)] = (n+1)[P_n(x) - P_{n+1}(x)]. \quad (12)$$

The corresponding identities for $Q_n(x)$ are

$$(1+x)[Q'_{n+1}(x) - Q'_n(x)] = (n+1)[Q_n(x) + Q_{n+1}(x)] \quad (13)$$

$$(1-x)[Q'_{n+1}(x) + Q'_n(x)] = (n+1)[Q_n(x) - Q_{n+1}(x)]. \quad (14)$$

Since

$$Q_n(-x) = (-1)^{n+1}Q_n(x) \quad (15)$$

it will be sufficient to prove either (13) or (14). The symmetry identity (15) is easy to check directly when $n=0$, and it then follows from

$$Q_1(x) = xQ_0(x) - 1$$

and (10). To prove (13), write

$$(1+x)[Q'_{n+1}(x) - Q'_n(x)] = \frac{(1+x)}{2} \int_{-1}^1 \frac{-P_{n+1}(t) + P_n(t)}{(x-t)^2} dt$$

$$\begin{aligned}
&= \frac{(1+x)}{2} \int_{-1}^1 [-P_{n+1}(t) + P_n(t)] \frac{1}{x-t} dt \\
&= \frac{(1+x)}{2} \left[\frac{P_n(t) - P_{n+1}(t)}{x-t} \right]_{-1}^1 + \frac{(1+x)}{2} \int_{-1}^1 \frac{P'_{n+1}(t) - P'_n(t)}{x-t} dt \\
&= (-1)^{n+1} + \frac{1}{2} \int_{-1}^1 (x-t) \frac{[P'_{n+1}(t) - P'_n(t)]}{x-t} dt \\
&\quad + \frac{1}{2} \int_{-1}^1 \frac{(1+t)[P'_{n+1}(t) - P'_n(t)]}{x-t} dt \\
&= (-1)^{n+1} + \frac{P_{n+1}(t) - P_n(t)}{2} \Big|_{-1}^1 + \frac{(n+1)}{2} \int_{-1}^1 \frac{[P_n(t) + P_{n+1}(t)]}{x-t} dt \\
&= (n+1)[Q_n(x) + Q_{n+1}(x)].
\end{aligned}$$

This proves (13) for x off $-1 \leq x \leq 1$, and for $-1 < x < 1$ it then follows by averaging via (6).

2. Zeros and maxima of $Q_n(x)$

Many properties of the Legendre polynomial continue to hold for the Legendre function $Q_n(x)$, but some are changed slightly. While the differential equation (1), and the differential difference relations (13) and (14) are exactly the same, the parity is opposite so the symmetry (15) is changed from

$$P_n(-x) = (-1)^n P_n(x) \quad (16)$$

and the recurrence relation (10) only holds for $n = 1, 2, \dots$, instead of for $n = 0, 1, \dots$. Another fact that changes slightly is the number of zeros in $-1 < x < 1$. $P_n(x)$ has n simple zeros, and the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other. Clearly $Q_0(x)$ vanishes to first order when $x = 0$, and since $Q_1(x)$ is convex and vanishes at least once between 0 and each of ± 1 , it has two simple zeros in $-1 < x < 1$. We will now show by induction that $Q_n(x)$ has $n+1$ zeros in $-1 < x < 1$. One standard way of doing this is to use what is called the Christoffel–Darboux formula in orthogonal polynomials. In the present setting, consider

$$D_n(x, y) = \sum_{k=1}^n (2k+1) Q_k(x) Q_k(y). \quad (17)$$

Then

$$\begin{aligned}
(x-y)D_n(x, y) &= Q_0(x)Q_1(y) - Q_1(x)Q_0(y) \\
&\quad + (n+1)[Q_{n+1}(x)Q_n(y) - Q_n(x)Q_{n+1}(y)] \\
&= -(x-y)Q_0(x)Q_0(y) + [Q_0(y) - Q_0(x)] \\
&\quad + (n+1)[Q_{n+1}(x)Q_n(y) - Q_n(x)Q_{n+1}(y)].
\end{aligned}$$

This gives

$$\begin{aligned}
\sum_{k=0}^n (2k+1) Q_k(x) Q_k(y) &= \frac{Q_0(y) - Q_0(x)}{x-y} \\
&\quad + (n+1) \left[\frac{Q_{n+1}(x)Q_n(y) - Q_n(x)Q_{n+1}(y)}{x-y} \right].
\end{aligned}$$

Next, letting $y \rightarrow x$ gives

$$\sum_{k=0}^n (2k+1)[Q_k(x)]^2 = -Q'_0(x) + (n+1)[Q'_{n+1}(x)Q_n(x) - Q'_n(x)Q_{n+1}(x)]. \quad (19)$$

But

$$Q'_0(x) = \frac{1}{2(1+x)} + \frac{1}{2(1-x)} = \frac{1}{1-x^2}$$

so

$$\frac{1}{1-x^2} + \sum_{k=0}^n (2k+1)[Q_k(x)]^2 = (n+1)[Q'_{n+1}(x)Q_n(x) - Q'_n(x)Q_{n+1}(x)]. \quad (20)$$

The left hand side is positive for $-1 < x < 1$, so the standard argument now works to show that between each pair of adjacent zeros of $Q_n(x)$ there is a zero of $Q_{n+1}(x)$, and between each pair of adjacent zeros of $Q_{n+1}(x)$ there is a zero of $Q_n(x)$. The arguments are the same, so only one will be given. Let $x_{k,n}$ denote the zeros of $Q_n(x)$ on $(-1, 1)$ with $x_{k+1,n} < x_{k,n}$ being adjacent zeros. The zeros are simple because $Q_n(x)$ satisfies (1). Let $R_n(x)$ denote the left hand side of (20). Since $R_n(x) > 0$, $-1 < x < 1$,

$$-R_n(x_{k,n}) = (n+1)Q'_n(x_{k,n})Q_{n+1}(x_{k,n}) < 0$$

and

$$-R_n(x_{k+1,n}) = (n+1)Q'_n(x_{k+1,n})Q_{n+1}(x_{k+1,n}) < 0.$$

Therefore

$$Q'_n(x_{k,n})Q'_n(x_{k+1,n})Q_{n+1}(x_{k,n})Q_{n+1}(x_{k+1,n}) > 0.$$

But $x_{k,n}$ and $x_{k+1,n}$ are adjacent simple zeros of $Q_n(x)$, so

$$Q'_n(x_{k,n})Q'_n(x_{k+1,n}) < 0,$$

which implies that

$$Q_{n+1}(x_{k,n})Q_{n+1}(x_{k+1,n}) < 0.$$

Thus $Q_{n+1}(x)$ has a zero between $x_{k+1,n}$ and $x_{k,n}$. This implies that $Q_{n+1}(x)$ has at least n zeros on $-1 < x < 1$. To show that there are at least $n+2$; and so exactly $n+2$, for more would imply $Q_n(x)$ has more than $n+1$ zeros on $-1 < x < 1$; it is sufficient to show that $Q_{n+1}(x)$ has a zero between $x_{1,n}$ and 1. If not, then $Q_{n+1}(x_{1,n}) > 0$. The graph of $Q_n(x)$, $n=0, 1, \dots, 5$ in [3, figure 59, p. 110] suggests that not only is $Q_{n+1}(x_{1,n}) < 0$, but that $Q_{n+1}(x) < Q_n(x)$ for $x_{1,n} < x < 1$. This can be proven by induction from (10). For it can be rewritten as

$$(n+1)[Q_n(x) - Q_{n+1}(x)] = n[Q_{n-1}(x) - Q_n(x)] + (1-x)(2n+1)Q_n(x).$$

Then for $x_{1,n-1} < x_{1,n} < x < 1$, we have

$$Q_{n-1}(x) - Q_n(x) > 0$$

by the induction assumption, and $Q_n(x) > 0$ since $Q_n(x) \rightarrow +\infty$ as $x \rightarrow 1^-$ from (3). This implies that $Q_n(x) > Q_{n+1}(x)$, $x_{1,n} < x < 1$, which is what we wanted to prove. Thus $x_{1,n} < x_{1,n+1}$, and so

$$x_{k+1,n+1} < x_{k,n} < x_{k,n+1}, \quad k = 1, 2, \dots, n+1 \quad (21)$$

$$Q'_n(y_{k,n}) = 0. \quad (22)$$

There is at least one such point in each interval $x_{k+1,n} < x < x_{k,n}$, and in fact there is exactly one, and none in $x_{1,n} < x < 1$. For, from the differential equation (1)

$$(1-x^2)y'(x) = (1-x_{k+1,n}^2)y'(x_{k+1,n}) - n(n+1) \int_{x_{k+1,n}}^x y(t) dt \quad (23)$$

so $(1-x^2)y'(x)$ is a monotone function for $x_{k+1,n} < x < x_{k,n}$, and so can vanish at most once. This takes care of the case $x_{k+1,n} < x < x_{k,n}$. For $x_{1,n} < x < 1$, $Q'_n(x)$ has no zeros since $(1-x^2)Q'_n(x)$ is monotone in x from (23), $Q'_n(x)$ is positive at $x=x_{1,n}$, and

$$\lim_{x \rightarrow 1} (1-x^2)Q'_n(x) = 1. \quad (24)$$

To show that

$$y_{k+1,n+1} < y_{k,n} < y_{k,n+1} \quad (25)$$

one argues in the same way as in the proof of (21). This argument is given for general orthogonal polynomials in [1]. Formula (20) is all that is needed, so this argument will not be included here.

With all of this as background we can now prove Theorem 1. If

$$\mu_{k,n} = |Q_n(y_{k,n})| \quad (26)$$

denotes the absolute values of the relative extrema of $Q_n(x)$ on $-1 < x < 1$, then

$$\mu_{k+1,n} < \mu_{k,n}, \quad k = 1, 2, \dots, \left[\frac{n}{2} \right] \quad (27)$$

and

$$\mu_{k,n+1} < \mu_{k,n}, \quad k = 1, \dots, n, \quad n = 1, 2, \dots \quad (28)$$

Proof. The proof of (27) is an exact copy of the same proof for Legendre polynomials, [5, Chapter VII]. Set

$$f(x) = [Q_n(x)]^2 + \frac{(1-x^2)[Q'_n(x)]^2}{n(n+1)}.$$

Then

$$\begin{aligned} f'(x) &= \frac{2Q'_n(x)}{n(n+1)} [(1-x^2)Q''_n(x) - xQ'_n(x) + n(n+1)Q_n(x)] \\ &= \frac{2x[Q'_n(x)]^2}{n(n+1)}. \end{aligned}$$

Thus $f(x)$ is increasing for $0 < x < 1$, and since

$$f(y_{k,n}) = \mu_{k,n}^2$$

this implies (27).

The proof of (28) is also a copy of one of the proofs of the same inequalities for Legendre polynomials. See Szegő [6]. Multiply (13) and (14) together to obtain

$$(1 - x^2)[[Q'_{n+1}(x)]^2 - [Q'_n(x)]^2] = (n+1)^2[[Q_n(x)]^2 - [Q_{n+1}(x)]^2]. \quad (29)$$

Set

$$\begin{aligned} f(x) &= (1 - x^2)[Q'_n(x)]^2 + (n+1)^2[Q_n(x)]^2 \\ &= (1 - x^2)[Q'_{n+1}(x)]^2 + (n+1)^2[Q_{n+1}(x)]^2. \end{aligned} \quad (30)$$

Then

$$\begin{aligned} f'(x) &= 2Q'_n(x)[(1 - x^2)Q''_n(x) - xQ'_n(x) + (n+1)^2Q_n(x)] \\ &= 2Q'_n(x)[xQ'_n(x) + (n+1)Q_n(x)] \end{aligned} \quad (31)$$

from the first equality in (30), and from the second

$$\begin{aligned} f'(x) &= 2Q'_{n+1}(x)[(1 - x^2)Q''_{n+1}(x) - xQ'_{n+1}(x) + (n+1)^2Q_{n+1}(x)] \\ &= 2Q'_{n+1}(x)[xQ'_{n+1}(x) - (n+1)Q_{n+1}(x)]. \end{aligned} \quad (32)$$

These two strongly suggest that

$$xQ'_n(x) + (n+1)Q_n(x) = c_n Q'_{n+1}(x) \quad (33)$$

and

$$xQ'_{n+1}(x) - (n+1)Q_{n+1}(x) = c_n Q'_n(x) \quad (34)$$

for a constant c_n . To see this add (13) and (14) to obtain

$$2Q'_{n+1}(x) - 2xQ'_n(x) = 2(n+1)Q_n(x),$$

which is (33) with $c_n = 1$. Subtracting gives (34) with $c_n = 1$. Thus

$$f'(x) = 2Q'_n(x)Q'_{n+1}(x).$$

Since

$$f(y_{k,n}) = \mu_{k,n}^2$$

and

$$f(y_{k,n+1}) = \mu_{k,n+1}^2,$$

to prove (28) it is sufficient to show that $f'(x) < 0$, $y_{k,n} < x < y_{k,n+1}$. But this is clear from the facts proven above about the zeros of $Q'_n(x)$.

Here is one consequence of Theorem 1.

COROLLARY 2.

If $0 < x < 1$, then $Q_n(x) \geq -1$ with equality only when $n = 1$ and $x = 0$.

Proof. For each fixed n and for $0 \leq x < 1$,

$$Q_n(x) > -\mu_{1,n}$$

by (27), and $-\mu_{1,n} > -\mu_{1,1}$ if $n > 1$ by (28).

When looking at either of the proofs of the monotonicity of the extrema of $P_n(x)$ as a function of n (given in [6] and [1]), it seems that the normalization of Legendre

polynomials by

$$P_n(1) = 1 \quad (35)$$

plays a central role. Since $Q_n(x) \rightarrow \infty$ as $x \rightarrow 1^-$, it is not clear that the normalization used above plays the same role for $Q_n(x)$ that (35) did for Legendre polynomials, but it does. There are two different aspects of this. One would like to have

$$\lim_{x \rightarrow 1^-} Q_n(x) - Q_{n+1}(x) = 0.$$

It is not true, but it almost is. For

$$Q_n(x) - Q_{n+1}(x) = Q_0(x)[P_n(x) - P_{n+1}(x)] + W_n(x) - W_{n-1}(x),$$

and

$$\begin{aligned} \lim_{x \rightarrow 1^-} Q_0(x)[P_n(x) - P_{n+1}(x)] &= \lim_{x \rightarrow 1} \frac{(1-x)}{2} \left[\log \frac{1+x}{1-x} \right] \frac{P_n(x) - P_{n+1}(x)}{1-x} \\ &= 0[P'_{n+1}(1) - P'_n(1)] = 0. \end{aligned}$$

However

$$\begin{aligned} W_n(1) - W_{n-1}(1) &= \frac{1}{2} \int_{-1}^1 \frac{1 - P_{n+1}(t)}{1-t} dt - \frac{1}{2} \int_{-1}^1 \frac{1 - P_n(t)}{1-t} dt \\ &= \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k} = \frac{1}{n+1}, \end{aligned}$$

so

$$\lim_{x \rightarrow 1^-} Q_n(x) - Q_{n-1}(x) = \frac{1}{n+1}. \quad (36)$$

The identity

$$\int_{-1}^1 \frac{1 - P_n(t)}{1-t} dt = 2 \left(\sum_{k=1}^n \frac{1}{k} \right) \quad (37)$$

can be proven in many ways. One is to prove it by induction using (9). My favorite way is given in [2]. One first evaluates

$$\int_{-1}^1 P_n(t)(1-t)^{-\gamma} dt$$

for $\gamma < 1$, which can be done by integration by parts using the Rodrigues formula, then gets

$$\int_{-1}^1 \frac{1 - P_n(t)}{(1-t)^\gamma} dt$$

and lets $\gamma \rightarrow 1^-$.

There is a better way of seeing that the normalization of $Q_n(x)$ is the right one (at least up to a constant independent of n). This comes from the asymptotic formulas

for $P_n(\cos \theta)$ and $Q_n(\cos \theta)$ for θ in $0 < \theta \leq \pi - \varepsilon$, $\varepsilon > 0$. Hilb's formula is

$$P_n(\cos \theta) = \left(\frac{\theta}{\sin \theta} \right)^{1/2} J_0 \left(\left(n + \frac{1}{2} \right) \theta \right) + O(n^{-3/2}) \quad (38)$$

uniformly for $0 \leq \theta < \pi - \varepsilon$, $\varepsilon > 0$. See Szegő [5, Theorem 8.21.6].

For $Q_n(\cos \theta)$ the corresponding formula is

$$Q_n(\cos \theta) = -\frac{\pi}{2} \left(\frac{\theta}{\sin \theta} \right)^{1/2} Y_0 \left(\left(n + \frac{1}{2} \right) \theta \right) + O(n^{-3/2})$$

uniformly for $0 < \theta \leq \pi - \varepsilon$, $\varepsilon > 0$. See Olver [4, (13.08) and Exercise 13.3].

Thus

$$\lim_{n \rightarrow \infty} \sigma_{k,n} = -\frac{\pi}{2} Y_0(y_{1,k}) \quad (39)$$

and

$$\lim_{n \rightarrow \infty} v_{k,n} = J_0(j_{1,k}). \quad (40)$$

$Y_0(x)$ is the usual second solution to the Bessel differential equation

$$x^2 y'' + xy' + x^2 y = 0,$$

$J_0(x)$ is the usual first solution, $y_{1,k}$ and $j_{1,k}$ are the zeros of their derivatives and $v_{k,n}$ is $P_n(t_{k,n})$ with $P'_n(t_{k,n}) = 0$, $t_{k+1,n} < t_{k,n}$, and $\sigma_{k,n}$ is $Q_n(y_{k,n})$, i.e. the same as $\mu_{k,n}$ when $\sigma_{k,n} > 0$, but $-\mu_{k,n}$ where $\sigma_{k,n} < 0$. Thus up to the factor $-\pi/2$ these are analogous. From this point of view it would be preferable to renormalize $Q_n(x)$, but it is probably too late to do so.

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Multipliers for the Weyl transform and Laguerre expansions

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Abstract. Let P_k denote the projection of $L^2(\mathbb{R}^n)$ onto the k th eigenspace of the operator $(-\Delta + |x|^2)$ and $S_N^\alpha = (1/A_N^\alpha) \sum_{k=0}^N A_{N-k}^\alpha P_k$. We study the multiplier transform T_N^α for the Weyl transform W defined by $W(T_N^\alpha f) = S_N^\alpha W(f)$. Applications to Laguerre expansions are given.

Keywords. Weyl transform; multipliers; twisted convolution; oscillatory singular integrals; Laguerre expansions; summability.

1. Introduction

Let $\hat{f}$ denote the Fourier transform of the function f and let $B = \{\xi : |\xi| \leq 1\}$ be the unit ball in $\mathbb{R}^n$. For $\alpha > 0$ define the multiplier operator

$$(T_\alpha f)^\wedge(\xi) = (1 - |\xi|^2)^\alpha \chi_B(\xi) \hat{f}(\xi). \quad (1)$$

It is well known that these operators are unbounded on $L^p(\mathbb{R}^n)$ for $p \leq (2n/(n+1+2\alpha))$ or $p \geq (2n/(n-1-2\alpha))$ for $0 < \alpha \leq ((n-1)/2)$. If $\alpha > ((n-1)/2)$ then T_α is bounded on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. When $\alpha = 0$ a celebrated theorem of Fefferman asserts that T_0 is bounded only on $L^2(\mathbb{R}^n)$. But if we consider only radial functions it is known that T_α is bounded for $(2n/(n+1+2\alpha)) < p < (2n/(n-1-2\alpha))$ if $0 \leq \alpha \leq ((n-1)/2)$.

In this paper we consider a similar problem for the Weyl transform. General multipliers for the Weyl transform have been studied by Mauceri [5]. Here we are interested in multipliers of the form

$$S_N^\alpha = \frac{1}{A_N^\alpha} \sum_{k=0}^N A_{N-k}^\alpha P_k \quad (2)$$

where P_k are the projections onto the k th eigenspace of the Hermite operator $H = -\Delta + |x|^2$. We call such multipliers Cesaro multipliers. Let W denote the Weyl transform which takes functions f on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. We define the multiplier operators T_N^α by setting

$$W(T_N^\alpha f) = S_N^\alpha W(f). \quad (3)$$

We would like to know for what values of α these operators T_N^α will be uniformly

Theorem 1. i) When $\alpha > n - \frac{1}{2}$, T_N^α is bounded on $L^p(\mathbb{C}^n)$ for $1 \leq p \leq \infty$. ii) When $\alpha = n - \frac{1}{2}$, T_N^α is bounded on $L^p(\mathbb{C}^n)$ for $1 < p < \infty$. iii) When $0 < \alpha < n - \frac{1}{2}$, T_N^α cannot be bounded unless p lies in the interval $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$. iv) If we consider only radial functions then T_N^α is bounded on $L^p(\mathbb{C}^n)$ for $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$ when $0 \leq \alpha \leq n - \frac{1}{2}$.

The study of the Weyl multipliers naturally involves the study of expansions in terms of the Laguerre functions $L_N^{n-1}(\frac{1}{2}|z|^2)\exp(-\frac{1}{4}|z|^2)$. Let $v_N^2 = (N!/(N+n-1)!)$ and consider the functions

$$\varphi_N(r) = v_N L_N^{n-1}(\frac{1}{2}r^2)\exp(-\frac{1}{4}r^2). \quad (4)$$

These functions form an orthonormal system for $L^2(\mathbb{R}_+, r^{2n-1}dr)$. So one would like to study the summability of the series

$$f(r) = \sum_{N=0}^{\infty} R_N(f) L_N^{n-1}(\frac{1}{2}r^2)\exp(-\frac{1}{4}r^2) \quad (5)$$

where

$$R_N(f) = v_N^2 \int_0^{\infty} f(r) L_N^{n-1}(\frac{1}{2}r^2)\exp(-\frac{1}{4}r^2)r^{2n-1}dr.$$

If $C_N^\alpha f$ stand for the Cesaro means of order α of the above series then Theorem 1 can be interpreted as summability results for the Laguerre expansions in the space $L^p(\mathbb{R}_+, r^{2n-1}dr)$. Thus we have the following theorem.

Theorem 2. i) The partial sums $C_N f$ of the series (1.5) converges to f in the norm iff $(4n/(2n+1)) < p < (4n/(2n-1))$. ii) When $\alpha > n - \frac{1}{2}$, $C_N^\alpha f$ converges to f in the norm for all p , $1 \leq p \leq \infty$. iii) $\alpha = n - \frac{1}{2}$ is the critical index in the sense that when $\alpha < n - \frac{1}{2}$ there is an L^1 function f such that $C_N^\alpha f$ will not converge in the norm to f . iv) When $\alpha = n - \frac{1}{2}$, $C_N^\alpha f$ converges to f whenever $1 < p < \infty$.

This paper is arranged as follows. In the next section we collect relevant facts about the Weyl transform and express the operators T_N^α as twisted convolution operators. Using that representation one easily proves that they are bounded on $L^p(\mathbb{C}^n)$ when $\alpha > n - \frac{1}{2}$. In §3 we consider the case $\alpha = n - \frac{1}{2}$. In §4, we consider radial functions from which the results about the Laguerre series (Theorem 2) will be deduced.

2. Preliminaries

We first recall the definition of the twisted convolution on $\mathbb{C}^n$. Let $\omega(z, v) = \exp(-(i/2)\text{Im } z \cdot \bar{v})$ and let $dv d\bar{v}$ stand for the Lebesgue measure on $\mathbb{C}^n$. Then the product

$$fXg(z) = \int_{\mathbb{C}^n} f(z-v)g(v)\omega(z, v)dv d\bar{v} \quad (6)$$

is called the twisted convolution of f and g . Let $B(L^2(\mathbb{R}^n))$ stand for the algebra of bounded linear operators on $L^2(\mathbb{R}^n)$. To define the Weyl transform we consider the operator valued function $W(z)$ taking values in $B(L^2(\mathbb{R}^n))$ defined by

$$W(z)\varphi(\xi) = \exp\{ix \cdot (\frac{1}{2}y + \xi)\}\varphi(\xi + y) \quad (7)$$

where $z = x + iy \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$. The Weyl transform of a function f is defined by

$$W(f) = \int_{\mathbb{C}^n} f(z) W(z) dz d\bar{z}. \quad (8)$$

Observe that $W(f)$ is a bounded operator on $L^2(\mathbb{R}^n)$.

For results concerning the Weyl transform we refer to the paper of Mauceri [5]. The Weyl transform enjoys many properties of the Fourier transform. For example there is an inversion formula and also a Plancherel theorem. Moreover, Weyl transform of the twisted convolution is the product of the Weyl transforms, i.e.,

$$W(fXg) = W(f)W(g). \quad (9)$$

Given a bounded operator M on $L^2(\mathbb{R}^n)$, one can define a multiplier operator T_M by setting $W(T_M f) = MW(f)$. Conditions on M have been found in [5] to ensure that T_M is bounded on $L^p(\mathbb{C}^n)$. Here we are interested in multipliers which are special functions of the Hermite operator $H = -\Delta + |x|^2$. The n -dimensional Hermite functions $\{\Phi_\alpha\}$ are the eigenfunctions for the operator H . Let P_k denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha: |\alpha| = k\}$. Our multipliers are then

$$S_N^\alpha = \frac{1}{A_N^\alpha} \sum_{k=0}^N A_{N-k}^\alpha P_k \quad (10)$$

where A_N^α are the binomial coefficients $A_N^\alpha = \binom{N}{\alpha}$.

Let $L_N^{n-1}(z)$ denote the Laguerre polynomial of type $(n-1)$. Then Peetre [6] has shown that the projection P_N is the Weyl transform of the function $\varphi_N(z) = L_N^{n-1}(\frac{1}{2}|z|^2) \exp(-\frac{1}{4}|z|^2)$, i.e., we have

$$P_N = W(\varphi_N) \quad (11)$$

Therefore, S_N^α is the Weyl transform of the function $s_N^\alpha(z)$ where

$$s_N^\alpha(z) = \frac{1}{A_N^\alpha} \sum_{k=0}^N A_{N-k}^\alpha \varphi_k(z). \quad (12)$$

But in view of the formula (see [8])

$$L_N^{\alpha+\beta+1}(x) = \sum_{k=0}^N A_{N-k}^\alpha L_k^\beta(x) \quad (13)$$

we obtain

$$s_N^\alpha(z) = \frac{1}{A_N^\alpha} L_N^{\alpha+n}(\frac{1}{2}|z|^2) \exp(-\frac{1}{4}|z|^2). \quad (14)$$

Thus we have proved that

$$W(T_N^\alpha f) = W(s_N^\alpha)W(f) \quad (15)$$

or $T_N^\alpha f$ is given by

$$T_N^\alpha f = s_N^\alpha \times f. \quad (16)$$

We see that T_N^α is a twisted convolution operator.

Now it is an easy matter to prove T_N^α are uniformly bounded on $L^p(\mathbb{C}^n)$ for $1 \leq p \leq \infty$ when $\alpha > n - \frac{1}{2}$. For that purpose we make use of the following asymptotic estimates available for the Laguerre functions. If $\mathcal{L}_N^\delta(r)$ are the normalized Laguerre functions then the following estimates are known (see [4]).

$$|\mathcal{L}_N^\delta(r)| \leq C \begin{cases} (rv)^{\delta/2} & \text{if } 0 \leq r \leq \frac{1}{v} \\ (rv)^{-1/4} & \text{if } \frac{1}{v} < r \leq \frac{v}{2} \\ \{v(v^{1/3} + |v-r|)\}^{-1/4} & \text{if } \frac{v}{2} < r \leq \frac{3v}{2} \\ \exp(-\gamma r) & \text{if } \frac{3v}{2} < r \end{cases}$$

where we have set $v = 4N + 2\delta + 2$. In view of these estimates one can easily check that

$$\int_{\mathbb{C}^n} |s_N^\alpha(z)| dz d\bar{z} \leq C \quad (17)$$

provided $\alpha > n - \frac{1}{2}$. Hence the operators T_N^α are uniformly bounded on L^p , $1 \leq p \leq \infty$ when $\alpha > n - \frac{1}{2}$.

It is also easy to see that T_N^α are not uniformly bounded on $L^1(\mathbb{C}^n)$ when $\alpha < n - \frac{1}{2}$. To prove this we suppose that T_N^α are bounded on $L^1(\mathbb{C}^n)$ and express S_N as follows (see [1]).

$$S_N = \sum_{k=0}^N A_k^\alpha A_{N-k}^{-\alpha-1} S_k^\alpha. \quad (18)$$

From this it follows that if $T_N = T_N^0$,

$$\|T_N f\|_1 \leq CN^\alpha \|f\|_1. \quad (19)$$

Since T_N is the twisted convolution operator with the function

$$L_N^n(\tfrac{1}{2}|z|^2) \exp(-\tfrac{1}{4}|z|^2)$$

the above will imply that

$$\int_0^\infty |L_N^n(\tfrac{1}{2}r^2)| \exp(-\tfrac{1}{4}r^2) r^{2n-1} dr \leq CN^\alpha. \quad (20)$$

Now we need a lower bound for the left hand side of (20).

The following estimates have been proved in [4].

$$\begin{aligned} \|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_p &\sim N^{(1/p)-(1/2)-(\beta/2)} & \text{if } \beta < \frac{2}{p} - \frac{1}{2} \\ \|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_p &\sim N^{(\beta/2)-(1/p)} & \text{if } \beta > \frac{2}{p} - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} C_1 N^{n-(1/2)} &\leq \int_0^\infty |L_N^n(\tfrac{1}{2}r^2)| \exp(-\tfrac{1}{4}r^2) r^{2n-1} dr \\ &\leq CN^\alpha \end{aligned}$$

and this can hold for large N only if $\alpha > n - \frac{1}{2}$. This completes the proof of part (i) of Theorem 1.

3. Cesaro multipliers when $\alpha \equiv n - \frac{1}{2}$

In this section we will prove that T_N^α are uniformly bounded on L^p , $1 < p < \infty$ when $\alpha = n - \frac{1}{2}$. If we examine the estimates for the normalized Laguerre functions stated in the previous section it is easy to see that

$$\begin{aligned} \int_{|z| \leq N^{-1/2}} |s_N^\alpha(z)| dz d\bar{z} &\leq C \\ \int_{|z| > N^{1/2}} |s_N^\alpha(z)| dz d\bar{z} &\leq C \end{aligned} \quad (21)$$

even when $\alpha = n - \frac{1}{2}$. Here the constant C is independent of N . If $A(N)$ denote the set $\{z: N^{-1/2} \leq |z| \leq N^{1/2}\}$ and if $\chi_{A(N)}$ stand for the characteristic function of $A(N)$ then it is enough to consider twisted convolution with the kernel

$$s_N^\alpha(z) \chi_{A(N)}(z). \quad (22)$$

Again we split this kernel into two parts. First we consider

$$\tilde{s}_N^\alpha(z) = s_N^\alpha(z) \chi_{B(N)}(z) \quad (23)$$

where $B(N) = \{z: 1 \leq |z| \leq N^{1/2}\}$. When $\alpha = n - \frac{1}{2}$ we have the estimate

$$|\tilde{s}_N^\alpha(z)| \leq C |z|^{-2n} \chi_{B(N)}(z) \quad (24)$$

Now we will prove the following proposition.

PROPOSITION 3.1.

The operator $f \rightarrow \tilde{s}_N^\alpha \times f$ is bounded on L^p for $1 < p < \infty$.

Proof. Observe that the kernel $\tilde{s}_N^\alpha$ is in $L^p(\mathbb{C}^n)$ for all $p > 1$. The proposition is then an immediate consequence of the following proposition whose proof can be found in [5].

PROPOSITION 3.2

Suppose that the kernel k is in $L^{1+\varepsilon}(\mathbb{C}^n)$ for $0 < \varepsilon < \varepsilon_0$. Then the twisted convolution operator $Kf = k \times f$ is bounded on $L^p(\mathbb{C}^n)$ for $1 < p < \infty$.

Thus we are left with the kernel $s_N^\alpha(z) \chi_{D(N)}(z)$ where $D(N) = \{z: N^{-1/2} \leq |z| \leq 1\}$.

$$s_N^{\alpha}(z) = C|z|^{-2n} \cos \left\{ 2N^{1/2}|z| - (\alpha + n)\frac{\pi}{2} - \frac{\pi}{4} \right\} + N^{-1/2}|z|^{-2n-1}O(1).$$

Since the error term is uniformly integrable we need to consider the truncated convolution with the kernel $K_N^{\alpha}(z)$ given by

$$K_N^{\alpha}(z) = |z|^{-2n} \psi(N^{1/2}|z|) \chi_{D(N)}(z)$$

where we have set

$$\psi(\theta) = \cos \left\{ 2\theta - (\alpha + n)\frac{\pi}{2} - \frac{\pi}{4} \right\}.$$

PROPOSITION 3.3

The operator $f \rightarrow K_N^{\alpha} * f$ is bounded on L^p for $1 < p < \infty$.

Proof. The proof follows in several steps. First we reduce the problem to the case of ordinary convolution with the kernel $k_N^{\alpha}(z)$. This is done as follows.

Consider a kernel $K(x, y)$ which satisfies the estimates

$$|K(x, y)| \leq C|x - y|^{-2n}$$

and assume that the operator

$$Tf(x) = \int K(x, y)f(y)dy$$

is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Consider now the truncated operators T_{ε} with

$$T_{\varepsilon}f(x) = \int_{|x-y| \leq \varepsilon} K(x, y)f(y)dy$$

Then we have the following Lemma due to Ricci and Stein [7].

Lemma 3.1. T_{ε} is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$ and the norm of T_{ε} is independent of ε .

Let $P(x, y)$ be a real polynomial. Consider then the operator

$$\tilde{T}_1 f(x) = \int_{|x-y| \leq 1} K(x, y) \exp(iP(x, y))f(y)dy$$

where the kernel K is as above. Then we have the following result.

Lemma 3.2. The operator $\tilde{T}_1$ is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$ whenever T is bounded on $L^p(\mathbb{R}^{2n})$.

A proof of this lemma can be found in [7] and it uses Lemma 3.1 together with an induction argument on the degree of the polynomial $P(x, y)$. Since the two

Proposition 3.2 can be proved once we show that the ordinary convolution operator $K_N^\alpha * f$ is bounded on $L^p(\mathbb{R}^{2n})$, for $1 < p < \infty$.

In twisted convolution the polynomial $P(x, y)$ is of first degree in x and y . It is easy to see that the roles of T and $\tilde{T}_1$ can be reversed in Lemma 3.2. Since the twisted convolution with K_N^α is bounded on $L^2(\mathbb{C}^n)$ we immediately see that $K_N^\alpha * f$ is bounded on $L^2(\mathbb{R}^{2n})$. Hence in view of the Marcinkiewicz interpolation theorem it is enough to prove that $K_N^\alpha * f$ is weak type $(1, 1)$.

Further reduction is affected by rescaling. It is enough to consider convolution with the kernel

$$R_N(x) = |x|^{-2n} \psi(|x|) \chi_{E(N)}(x)$$

where $E(N) = \{x: 1 \leq |x| \leq N^{1/2}\}$. If we have a weak type $(1, 1)$ analogue of Lemma 3.1 for truncated operators, then the claim that $K_N^\alpha * f$ is of weak type $(1, 1)$ will follow from the following theorem due to Christ [3].

Theorem. Let $K(x) = |x|^{-2n} \psi(|x|) \chi_{\{|x| \geq 1\}}(x)$. Then convolution with $K(x)$ is of weak type $(1, 1)$.

So we are left with proving the following.

Lemma 3.3. Let T and T_ε be as in Lemma 3.1. Then if T is of weak type $(1, 1)$, so is T_ε .

Proof. By scale invariance we can assume that $\varepsilon = 1$. Let us write $\mathbb{R}^{2n}$ as a union of disjoint cubes Q_j having centres at y_j and each Q_j having side length $\frac{1}{4}$ with sides parallel to the coordinate axes. We will show that

$$|\{x \in Q_j : |T_1 f(x)| > \lambda\}| \leq \frac{c}{\lambda} \int_{5Q_j} |f(y)| dy \quad (29)$$

where $|E|$ is the Lebesgue measure of E . The Lemma will then follow by summing over j noticing that the doubles of the cubes have bounded overlap property.

Let us write $f = f_1 + f_2 + f_3$ where $f_1 = f \chi_{2Q_j}$ and $f_2 = f \chi_{5Q_j \setminus 2Q_j}$. When $x \in Q_j$ and $y \in 2Q_j$ we have $|x - y| \leq 1$ and hence for x in Q_j

$$T_1 f_1(x) = \int K(x, y) f_1(y) dy = T f_1(x).$$

Since T is of weak type $(1, 1)$

$$\left| \left\{ x \in Q_j : |T_1 f_1(x)| > \frac{\lambda}{3} \right\} \right| \leq \frac{c}{\lambda} \int_{2Q_j} |f(y)| dy. \quad (30)$$

When $x \in Q_j$ and $y \in 5Q_j \setminus 2Q_j$ we have $|x - y| \geq \frac{1}{4}$. Since the kernel satisfies the estimate $K(x, y) \leq C|x - y|^{-2n}$ it is integrable when $\frac{1}{4} \leq |x - y| \leq 1$. Hence

$$T_1 f_2(x) = \int_{\frac{1}{4} \leq |x - y| \leq 1} K(x, y) f_2(y) dy \quad (31)$$

is bounded on L^1 and in particular

$$\left| \left\{ x \in Q_j : |T_1 f_2(x)| > \frac{\lambda}{3} \right\} \right| \leq \frac{c}{\lambda} \int_{5Q_j \setminus 2Q_j} |f(y)| dy$$

Finally, when $x \in Q_j$, $y \in \mathbb{R}^{2n} \setminus 5Q_j$ we have $|x - y| > 1$ and hence $T_1 f_3(x) = 0$. Thus we have proved that

$$|\{x \in Q_j : |T_1 f(x)| > \lambda\}| \leq \frac{c}{\lambda} \int_{5Q_j} |f(y)| dy \quad (32)$$

This completes the proof of Lemma 3.3.

4. Radial functions and Laguerre expansions

When f is a radial function the Weyl transform $W(f)$ reduces to the Laguerre transform and we have the formula

$$W(f) = \sum_{N=0}^{\infty} R_N(f) P_N \quad (33)$$

where $R_N(f)$ are defined by

$$R_N(f) = \frac{N!}{(N+n-1)!} \int_0^{\infty} f(r) L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2) r^{2n-1} dr. \quad (34)$$

Therefore, for radial functions

$$T_N^{\alpha} f = \frac{1}{A_N^{\alpha}} \sum_{k=0}^N A_{N-k}^{\alpha} R_k(f) L_k^{n-1}(\tfrac{1}{2}|z|^2) \exp(-\tfrac{1}{4}|z|^2). \quad (35)$$

Thus the study of the operators T_N^{α} reduces to the study of the Cesaro means of order α of the series

$$f(r) = \sum_{k=0}^{\infty} R_k(f) L_k^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2). \quad (36)$$

Let us set $v_N^2 = (N!/(N+n-1)!)$ and define

$$\varphi_N^{n-1}(r) = v_N L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2) r^{n-1}.$$

Let $s_N^{\alpha}(r, s)$ be the kernel

$$s_N^{\alpha}(r, s) = \frac{1}{A_N^{\alpha}} \sum_{k=0}^N A_{N-k}^{\alpha} \varphi_k^{n-1}(r) \varphi_k^{n-1}(s). \quad (37)$$

Then it is clear that $T_N^{\alpha} f(s)$ is given by

$$T_N^{\alpha} f(s) = s^{-(n-1)} \int_0^{\infty} s_N^{\alpha}(r, s) f(r) r^n dr \quad (38)$$

We will prove the following proposition.

Assume that $0 \leq \alpha \leq n - \frac{1}{2}$, $f \in L^p(\mathbb{R}_+, r^{2n-1} dr)$ and p satisfies $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$. Then

$$\int_0^\infty |T_N^\alpha f(s)|^p s^{2n-1} ds \leq C \int_0^\infty |f(r)|^p r^{2n-1} dr$$

with a constant C independent of N .

Proof. In view of Stein's interpolation theorem for analytic families of operators and the L^1 result for $\alpha > n - \frac{1}{2}$ it is enough to consider the case $\alpha=0$. So let T_N be the operator defined by

$$T_N f(s) = s^{-(n-1)} \int_0^\infty s_N(r, s) f(r) r^n dr \quad (39)$$

where

$$s_N(r, s) = \sum_{k=0}^N \varphi_k^{n-1}(r) \varphi_k^{n-1}(s). \quad (40)$$

The case $n=1$ was settled by Askey and Wainger in [2] because then we are considering the convergence of the usual Laguerre series. The higher dimensional case is not very different except that certain calculations become more tedious. We will not give a complete detailed proof. Instead we will bring out the analogy between the one dimensional and the higher dimensional case and then refer to the paper of Askey–Wainger.

We are interested in establishing the inequality

$$\int_0^\infty |T_N f(s)|^p s^{2n-1} ds \leq C \int_0^\infty |f(r)|^p r^{2n-1} dr \quad (41)$$

for $(4n/(2n+1)) < p < (4n/(2n-1))$. Let us make the following reductions. Define $g(r) = f(r^{1/2n})$ and

$$K_N(r, s) = s^{-[(n-1)/2n]} s_N(r^{1/2n}, s^{1/2n}) r^{-[(n-1)/2n]}$$

and let K_N be the operator with kernel $K_N(r, s)$. Then we need to establish the inequality

$$\int_0^\infty |K_N g(s)|^p ds \leq C \int_0^\infty |g(r)|^p dr. \quad (42)$$

As in Askey–Wainger we use Pollard's device of considering

$$s_N + s_{N+1} = 2s_N + \varphi_{N+1}^{n-1}(r) \varphi_{N+1}^{n-1}(s). \quad (43)$$

This leads to the formula

$$s_N(r, s) = \frac{N \varphi_{N+1}^{n-1}(s)}{r^2 - s^2} \{ \varphi_{N+2}^{n-1}(r) - \varphi_N^{n-1}(r) \} + \text{more terms.}$$

Consequently the kernel $K_N(r, s)$ is given by

$$K_N(r, s) = \frac{N(rs)^{-(n-1)/2n} \varphi_{N+1}^{n-1}(s^{1/2n})}{r^{1/n} - s^{1/n}} \{ \varphi_{N+1}^{n-1}(r^{1/2n}) - \varphi_N^{n-1}(r^{1/2n}) \} + \text{more terms}$$

We just consider the first term which defines the operator

$$A_N g(s) = s^{-(n-1)/2n} N \varphi_{N+1}^{n-1}(s^{1/2n}) \int_0^\infty \frac{\varphi_{N+1}^{n-1}(r^{1/2n}) - \varphi_N^{n-1}(r^{1/2n})}{r^{1/n} - s^{1/n}} g(r) r^{-(n-1)/2n} dr.$$

Using the identity

$$(sr)^{-(n-1)/2n} (r^{1/n} - s^{1/n})^{-1} = (r-s)^{-1} \sum_{k=1}^n (r^{1/n})^{(n+1)/2-k} (s^{1/n})^{k-(n+1)/2}$$

we are led to consider operators of the form

$$A_N^k g(s) = (s^{1/n})^{k-(n+1)/2} N \varphi_{N+1}^{n-1}(s^{1/2n}) \times \int_0^\infty \frac{\varphi_{N+1}^{n-1}(r^{1/2n}) - \varphi_N^{n-1}(r^{1/2n})}{r-s} (r^{1/n})^{(n+1)/2-k} g(r) dr.$$

Let us now briefly recall how Askey and Wainger established the inequality (4) in the one dimensional case. Dividing the r -range of integration and the s -range of integration into 6 parts one is led to consider 36 integrals. Using asymptotic estimates for Laguerre functions some of the integrals can be estimated directly. The remaining integrals can be expressed in one of the following three forms:

$$\begin{aligned} & \varphi(s, N) \int \frac{f(r)}{r-s} \psi(r, N) dr \\ & |s|^{-1/4} \varphi(s, N) \int \frac{f(r)}{r-s} \psi(r, N) |r|^{1/4} dr \\ & |s-v|^{-1/4} \varphi(s, N) \int \frac{f(r)}{r-s} \psi(r, N) |r-v|^{1/4} dr. \end{aligned}$$

To estimate integrals of the above forms they used a weighted inequality for Hilbert transform and a weighted Hardy's inequality. Let us recall what these are. If we define

$$F_{1/4}(x) = x^{-5/4} \int_0^x f(t) t^{1/4} dt$$

then for $\frac{4}{3} < p < 4$ one has

The weighted Hilbert transform is defined by

$$\tilde{f}_{1/4}(x) = \lim_{\varepsilon \rightarrow 0} |x|^{-1/4} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} |y|^{1/4} dy.$$

For this Hilbert transform one has

$$\|\tilde{f}_{1/4}\|_p \leq C \|f\|_p$$

provided $\frac{4}{3} < p < 4$. The condition $\frac{4}{3} < p < 4$ is imposed by the fact that the weight $|x|^{p/4}$ is in the A_p class of Muckenhoupt iff $p > \frac{4}{3}$.

One can do the same thing in the case when $n > 1$. Instead of the weights $|r|^{\pm(p/4)}$, this time we have to consider the weights $|r|^{\pm\alpha p}$ where $0 < \alpha < 1$. These weights will be in A_p class iff p lies between $1/\alpha$ and $1/(1-\alpha)$ (For facts about the A_p class we refer to the book [9] of Torchinsky).

Using asymptotic estimates for the functions $\varphi_N^{n-1}(r)$ as in [2] one can establish the inequality (42) by appealing to the weighted Hilbert transform and Hardy's inequality. Because of various weights of the form $|r|^{\pm\alpha p}$ the condition $(4n/(2n+1)) < p < (4n/(2n-1))$ for the validity of (42) is naturally imposed. We omit the details.

Finally it remains to prove the necessity of the condition $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$ for the boundedness of T_N^α for radial functions. That is the content of the next proposition.

PROPOSITION 4.2

The inequality (41) holds only if p satisfies the condition $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$.

Proof. As we know that T_N^α are bounded on L^p radial functions when $(4n/(2n+1)) < p < (4n/(2n-1))$ we consider a p satisfying $(4n/(2n-1)) < p \leq 4$. Then the conjugate exponent q will satisfy $q < (4n/(2n+1))$. Suppose that we have the inequality

$$\int_0^\infty |T_N^\alpha f(s)|^p s^{2n-1} ds \leq C \int_0^\infty |f(s)|^p s^{2n-1} ds.$$

Expressing the partial sums in terms of the Cesaro means of order α as in §2 we get

$$|R_N(f)| \left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1} dr \right)^{1/p} \leq CN^\alpha \|f\|_p.$$

By a proper choice of f the above will lead to the inequality

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1} dr \right)^{1/p} \cdot \left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^q r^{2n-1} dr \right)^{1/q} \leq CN^{\alpha+n-1}.$$

Again we make use of the norm estimates stated in §2

$$\begin{aligned} & \left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1} dr \right)^{1/p} \\ &= CN^{(n-1)/2} \left(\int_0^\infty |\mathcal{L}_N^{n-1}(r) r^{(n-1)((1/p)-(1/2))}|^p dr \right)^{1/p}. \end{aligned}$$

In the notation of §2 we have $\alpha + \beta = n - 2$, $\beta = 2(n-1)((1/2)-(1/p))$. Since $p > (4n/(2n-1))$ it is easy to see that $\beta > ((2/p)-(1/2))$. Therefore,

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1} dr \right)^{1/p} \geq CN^{(n-1)-(n/p)}.$$

For the other term as $q < (4n/(2n+1))$ we will have $\beta = 2(n-1)((1/2)-(1/p)) < ((2/q)-(1/2))$ and hence

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^q r^{2n-1} dr \right)^{1/q} \geq CN^{((n-1)/2)+n((1/q)-(1/2))}.$$

Hence we have the inequality

$$N^{n-1} N^{((n-1)/2)} N^{((n/2)-(n/p)+(n/q))} \leq CN^{\alpha+n-1}$$

This can hold for large N only if

$$n\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2} \leq \alpha$$

which after simplification becomes $p \leq (4n/(2n-1-2\alpha))$. Similarly one can show that the condition $p \geq (4n/(2n+1+2\alpha))$ is also necessary.

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On the inverse Laplace transform of the product of a general class of polynomials and the multivariable H -function

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Abstract. In this paper we evaluate the inverse Laplace transform of

$$\begin{aligned} & s^{-\eta}(s^{l_1} + \lambda_1)^{-\sigma}(s^{l_2} + \lambda_2)^{-\rho} \\ & \times S_n^m[xs^{-u}(s^{l_1} + \lambda_1)^{-v}(s^{l_2} + \lambda_2)^{-w}]S_n^{m'}[ys^{-u'}(s^{l_1} + \lambda_1)^{-v'}(s^{l_2} + \lambda_2)^{-w'}] \\ & \times H[z_1s^{-u_1}(s^{l_1} + \lambda_1)^{-v_1}(s^{l_2} + \lambda_2)^{-w_1}, \dots, z_rs^{-u_r}(s^{l_1} + \lambda_1)^{-v_r}(s^{l_2} + \lambda_2)^{-w_r}]. \end{aligned}$$

Due to the general nature of the multivariable H -function involved herein, the inverse Laplace transform of the product of a large number of special functions involving one or more variables, occurring frequently in the problems of theoretical physics and engineering sciences can be obtained as simple special cases of our main findings. For the sake of illustration, we obtain here the inverse Laplace transform of a product of the Hermite polynomials, the Jacobi polynomials and r different modified Bessel functions of the second kind. A theorem obtained by Srivastava and Singh [7] follows as a special case of our main result.

Keywords. The Laplace transform; the multivariable H -function; a general class of polynomials.

1. Introduction

We define the Laplace transform of the function $f(x)$ as

$$\phi(s) = L\{f(x); s\} = \int_0^\infty \exp(-sx)f(x)dx. \quad (1)$$

The function $f(x)$ is called the inverse Laplace transform of $\phi(s)$ and will be denoted by $L^{-1}\{\phi(s)\}$ in this paper.

Main result

$$\begin{aligned} & L^{-1}\{s^{-\eta}(s^{l_1} + \lambda_1)^{-\sigma}(s^{l_2} + \lambda_2)^{-\rho} \\ & \times S_n^m[xs^{-u}(s^{l_1} + \lambda_1)^{-v}(s^{l_2} + \lambda_2)^{-w}]S_n^{m'}[ys^{-u'}(s^{l_1} + \lambda_1)^{-v'}(s^{l_2} + \lambda_2)^{-w'}] \\ & \times H[z_1s^{-u_1}(s^{l_1} + \lambda_1)^{-v_1}(s^{l_2} + \lambda_2)^{-w_1}, \dots, z_rs^{-u_r}(s^{l_1} + \lambda_1)^{-v_r}(s^{l_2} + \lambda_2)^{-w_r}]\} \end{aligned}$$

where the symbols S_n'' and S_n''' occurring in the left hand side of (2) denote product of Srivastava's general class of polynomials [4, p. 1, eq.(1)] and the H-functions occurring in its left hand side and right hand side stand for the H-function of r and $r+2$ variables respectively [6] expressed with slightly changed notations in this paper [5, p.251, eq.(C.1)] and the following conditions are satisfied. The quantities $u, v, w, u', v', w', u_i, v_i, w_i, i=1, \dots, r$ are all positive, $\text{Re}(s) > 0$.

$$\begin{aligned} & \operatorname{Re}(\eta + l_1 \sigma + l_2 \rho) + \min_{1 \leq j \leq M^{(i)}} \left[\operatorname{Re}(u_i + v_i l_1 + w_i l_2) \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0, \\ & \sum_{j=1}^P \alpha_j^{(i)} + \sum_{j=1}^{P^{(i)}} \gamma_j^{(i)} - \sum_{j=1}^Q \beta_j^{(i)} - \sum_{j=1}^{Q^{(i)}} \delta_j^{(i)} - (u_i + v_i l_1 + w_i l_2) < 0, \quad i = 1, \dots, r; \\ & U_i = - \sum_{j=N+1}^P \alpha_j^{(i)} + \sum_{j=1}^{N^{(i)}} \gamma_j^{(i)} - \sum_{j=N^{(i)}+1}^{P^{(i)}} \gamma_j^{(i)} - \sum_{j=1}^Q \beta_j^{(i)} + \sum_{j=1}^{M^{(i)}} \delta_j^{(i)} - \sum_{j=M^{(i)}+1}^{Q^{(i)}} \delta_j^{(i)} \\ & \quad - [u_i + v_i(l_1 + 1) + w_i(l_2 + 1)] > 0, \\ & |\arg z_i| < (\frac{\pi}{2}) U_i \pi \text{ or } U_i = 0 \text{ and } z_i > 0, \quad i = 1, \dots, r; \\ & 0 < l_j < 1, |\arg \lambda_j| < (1 - l_j) \frac{\pi}{2}, j = 1, 2 \text{ or } l_1 = l_2 = 1 \text{ and } \lambda_1 > 0, \lambda_2 > 0. \end{aligned}$$

provided the conditions easily obtainable from (2) are satisfied. (ii) If we take $N = P = Q = 0$, $M^{(i)} = Q^{(i)} = 2$, $N^{(i)} = P^{(i)} = 0$, $d_1^{(i)} = \frac{1}{4} + (v_i/2)$, $d_2^{(i)} = \frac{1}{4} - (v_i/2)$, $\delta_1^{(i)} = \delta_2^{(i)} = u_i = 1$, $v_i = w_i = 0$ and replace z_i by $z_i/2$ in (3), we arrive at the following interesting result involving the product of r different modified Bessel functions of the second kind

$$\begin{aligned} & L^{-1} \left\{ s^{-(\eta + n/2 + r/2)} (s^{l_1} + \lambda_1)^{-\sigma} (s^{l_2} + \lambda_2)^{-\rho} H_n \left[\frac{\sqrt{s}}{2} \right] P_{n'}^{(\delta, \zeta)} \left[1 - \frac{2}{s} \right] \prod_{i=1}^r K_{v_i} \left[\frac{z_i}{s} \right] \right\} \\ &= 2^{-3r/2} \prod_{i=1}^r (z_i)^{-1/2} \frac{t^{\eta + l_1 \sigma + l_2 \rho - 1}}{\Gamma(\sigma) \Gamma(\rho)} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{k'=0}^{n'} \frac{(-n)_{2k} (-n')_{k'}}{k! k'!} \\ &\quad \times (-1)^k \binom{n' + \delta}{n'} \frac{(\delta + \zeta + n' + 1)_{k'}}{(\delta + 1)_{k'}} t^{k+k'} \\ &\quad \times H_{\substack{0, 0; 2, 0; \dots; 2, 0; 1, 1; 1 \\ 0, 1; 0, 2; \dots; 0, 2; 1, 1; 1, 1}} \left[\begin{matrix} z_1 t \\ \vdots \\ z_r t \\ z_1 t^{l_1} \\ \lambda_2 t^{l_2} \end{matrix} \middle| \begin{matrix} \dots \\ (1 - \eta - l_1 \sigma - l_2 \rho - k - k'; 1, \dots, 1, l_1 \\ \dots \\ (1 - \sigma, 1); (1 - \rho, 1) \end{matrix} \right] \\ &\quad \left(\frac{1}{4} \pm \frac{v_i}{2}, \frac{1}{2} \right); \dots; \left(\frac{1}{4} \pm \frac{v_r}{2}, \frac{1}{2} \right); (0, 1) \quad ; (0, 1) \end{aligned}$$

where

$$\operatorname{Re}(s) > 0, z_i > 0, \operatorname{Re}(\eta + l_1 \sigma + l_2 \rho \pm v_i + \frac{1}{2}) > 0, i = 1, \dots, r;$$

$$0 < l_j < 1, |\arg \lambda_j| < (1 - l_j) \frac{\pi}{2}, j = 1, 2 \text{ or } l_1 = l_2 = 1 \text{ and } \lambda_1 > 0, \lambda_2 > 0.$$

The inverse Laplace transforms of the product of numerous other polynomials (which are special cases of S_n^m and S_n^m) and special functions involving one or more variables (which are particular cases of the multivariable H -function) can also be obtained from our main result.

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Compact operators on a Banach space into the space of almost periodic functions

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Abstract. Let S be a topological semigroup and $AP(S)$ the space of continuous complex almost periodic functions on S . We obtain characterizations of compact and weakly compact operators from a Banach space X into $AP(S)$. For this we use the almost periodic compactification of S obtained through uniform spaces. For a bounded linear operator T from X into $AP(S)$, let T_s be the translate of T by s in S defined by $T_s(x) = (Tx)_s$. We define topologies on the space of bounded linear operators from X into $AP(S)$ and obtain the necessary and sufficient conditions for an operator T to be compact or weakly compact in terms of the uniform continuity of the map $s \rightarrow T_s$. If S is a Hausdorff topological semigroup, we also obtain characterizations of compact and weakly compact multipliers on $AP(S)$ in terms of the uniform continuity of the map $s \rightarrow \mu_s$, where μ_s denotes the unique vector measure corresponding to the operator T_s .

Keywords. Compact operators; almost periodic functions.

1. Introduction

Let S be a compact Hausdorff topological space and $C(S)$, the space of bounded continuous complex functions on S with uniform topology. Characterizations of compact and weakly compact linear operators from a Banach space X into $C(S)$ are given by Dunford and Schwartz [5, theorem VI.7.3, 7.7]. We consider the case in which S is further a topological semigroup with a uniform structure. We obtain the necessary and sufficient conditions for operators from X into $C(S)$ to be compact or weakly compact. Let $(S, \mathcal{U})$ be a compact Hausdorff topological semigroup, where the uniformity $\mathcal{U}$ is left translation invariant. (That is, for every entourage U in $\mathcal{U}$, $(s, t) \in U$ implies $(us, ut) \in U$ for all $u \in S$). Let H be a dense subsemigroup of $(S, \mathcal{U})$. For Banach spaces X and Y let $BL(X, Y)$ be the space of bounded linear operators from X into Y . When $X = Y$ we write $BL(X)$ for $BL(X, Y)$. For $T \in BL(X, C(S))$ and $s \in S$, define $T_s: X \rightarrow C(S)$ by $T_s(x) = (Tx)_s$, where $(Tx)_s$ is the right translate of Tx . As usual X^* denotes the topological dual of X .

In §3, it is proved that an operator $T: X \rightarrow C(S)$ is compact if and only if the map $s \rightarrow T_s$ from H into $BL(X, C(S))$ is uniformly continuous in the norm topology of $BL(X, C(S))$. We define a topology τ on $BL(X, C(S))$ and characterize weakly compact operators T in terms of the uniform continuity of the maps $s \rightarrow T_s$ from H into

$BL(X, C(S))$ with respect to the topology τ . In these results we consider the maps $s \rightarrow T_s$ for s in the dense subsemigroup H of S because this is convenient for its applications considered in §3.

For a topological semigroup S , let $AP(S)$ denote the space of continuous complex almost periodic functions on S . In §3, we characterize compact and weakly compact operators from a Banach space X into $AP(S)$. For this, we first construct the almost periodic compactification S^a of S by using uniform spaces. The space S^a is a compact Hausdorff topological semigroup and in fact coincides with the almost periodic compactification of S obtained by [4] using operators. It is well-known that the multiplication in S^a is jointly uniformly continuous. But what is essential for the results obtained here is the fact that the uniformity of S^a is left translation invariant. It is proved in [4] and [8] that the space $AP(S)$ is isometrically isomorphic to $C(S^a)$. But we give an independent proof of this fact (theorem 1.6). Theorems of §3 are then applied to the operators from X into $C(S^a)$ to obtain the desired characterizations.

Section 5 deals with the operators in $BL(C(S))$ where S is a compact Hausdorff topological semigroup with left translate invariant uniformity. Let $\mathcal{B}$ be the σ -field of Borel sets in S and $rca(S)$ the space of regular countably additive scalar valued measures defined on $\mathcal{B}$. For a Banach space X , let X^{**} denote the second dual of X and $F(S, \mathcal{B}, X^{**})$, the space of set functions μ on $\mathcal{B}$ into X^{**} such that $\mu(\cdot)x^* \in rca(S)$ for every $x^* \in X^*$, where $(\mu(\cdot)x^*)(E) = \mu(E)x^*$, $E \in \mathcal{B}$. Let $T \in BL(C(S), X)$ and μ be the unique set function which represents the operator T [5, theorem VI.7.2]. Then, proposition 4.3 shows that for each $s \in S$ the set function $\mu_s: \mathcal{B} \rightarrow X^{**}$ defined by $\mu_s(E) = \mu(Es)$ represents the operator $T_s \in BL(C(S), X)$ where $T_s(f) = T(f_s)$, $f \in C(S)$. In theorem 4.4 it is proved that a multiplier T in $BL(C(S))$ is compact if and only if the map $s \rightarrow \mu_s$ from a dense subsemigroup H of S into $F(S, \mathcal{B}, C(S)^{**})$ is uniformly continuous in the topology on $F(S, \mathcal{B}, C(S)^{**})$ defined in terms of semivariation. We also characterize weakly compact multipliers in $BL(C(S))$ in terms of the uniform continuity of the map $s \rightarrow \mu_s$ from H into $(F(S, \mathcal{B}, C(S)^{**}), \tau)$ (theorem 4.5). In this case the set function μ_s is in fact a vector measure defined on $\mathcal{B}$ and takes values in $C(S)$. Using these results and the almost periodic compactification of S , in theorems 4.6 and 4.7 we obtain the necessary and sufficient conditions for multipliers in $BL(AP(S))$ to be compact or weakly compact. If μ is a translation invariant vector measure then $\mu_s - \mu_t = 0$, where $\mu_s(E) = \mu(Es)$. Our results show that for vector measures corresponding to compact multipliers, $\mu_s - \mu_t$ is 'small' if s, t are 'near' to one another. This is thus a consequence of the assumption that the uniform structure on S is left translation invariant.

2. Preliminaries

We begin this section with the following definitions from [1].

DEFINITION 1.1

A set S is called a topological semigroup if S is a semigroup with identity e and if S has a Hausdorff topology such that the multiplication on S is separately continuous. That is, for each $t \in S$ the maps $s \rightarrow st$ and $s \rightarrow ts$ are continuous functions.

For f in $C(S)$ and $s \in S$, define the right translate f_s of f by $f_s(t) = f(ts)$, $t \in S$ and the orbit of f , $O_R(f) = \{f_s : s \in S\}$.

DEFINITION 1.3

A function f in $C(S)$ is right almost periodic if $O_R(f)$ is relatively compact in the norm topology of $C(S)$.

Left almost periodic functions are similarly defined. The set of right almost periodic functions coincides with the set of left almost periodic functions [7, p. 167]. We denote this common set by $AP(S)$ and call its functions almost periodic. For the proof of the following lemma, we refer [6, theorem 4.2].

Lemma 1.4. *The space $AP(S)$ is translation invariant norm closed linear subspace of $C(S)$ and hence a Banach space.*

To obtain results mentioned in the introduction we construct the almost periodic compactification of a topological semigroup as follows: Construction of S^a : Let S be a topological semigroup. Denote by $\tilde{S}$ the quotient structure of S determined by the equivalence relation, s is related to t if and only if $f_s = f_t$ for all f in $AP(S)$. Then $\tilde{S}$ is a semigroup with the multiplication $\tilde{s} \cdot \tilde{t} = \widetilde{st}$. For $\varepsilon > 0$ and f in $AP(S)$ define

$$U(\varepsilon, f) = \{(\tilde{s}, \tilde{t}) \in \tilde{S} \times \tilde{S} : \|f_s - f_t\| < \varepsilon \text{ for all } s \in \tilde{s}, t \in \tilde{t}\}.$$

Then the family of sets of the form $U(\varepsilon, f)$ forms a subbase for a uniformity, say $\tilde{\mathcal{U}}$, on $\tilde{S}$. We note that $\tilde{\mathcal{U}}$ is left translation invariant, since $\|f_{us} - f_{ut}\| \leq \|f_s - f_t\|$ for all $u \in S$. The proof of the following theorem is not difficult and hence it is omitted.

Theorem 1.5. *The space $(\tilde{S}, \tilde{\mathcal{U}})$ is a totally bounded Hausdorff topological semigroup and its completion, say S^a , is a compact Hausdorff topological semigroup on which the multiplication is jointly uniformly continuous.*

Theorem 1.6. *Let S be a topological semigroup. The homomorphism $\rho: S \rightarrow (\tilde{S}, \tilde{\mathcal{U}}) \subset S^a$ defined by $\rho(s) = \tilde{s}$ is continuous with $\rho(S)$ dense in S^a . The induced map $\tilde{\rho}: C(S^a) \rightarrow C(S)$ given by $\tilde{\rho}(\hat{f}) = \hat{f} \circ \rho$ is an isometric algebra isomorphism of $C(S^a)$ onto $AP(S)$.*

Proof. By [1, theorem 1.7(iii)] for each f in $AP(S)$ the map $s \rightarrow f_s$ is norm continuous from S into $AP(S)$. Therefore, it follows that ρ is continuous. Also, since $\rho(S) = (\tilde{S}, \tilde{\mathcal{U}})$, a dense subspace of S^a , $\tilde{\rho}$ is one-one and an isometry. Moreover $\tilde{\rho}$ preserves the ordinary multiplication of functions, so $AP(S)$ is an algebra and $\tilde{\rho}$ is an algebra isomorphism. Therefore it remains to prove that $\tilde{\rho}$ maps $C(S^a)$ onto $AP(S)$. By [1, theorem 1.8 (ii)], $\tilde{\rho}[C(S^a)] \subset AP(S)$. To prove the reverse inclusion, let $f \in AP(S)$. Define $\tilde{f}: (\tilde{S}, \tilde{\mathcal{U}}) \rightarrow \mathbb{C}$ by $\tilde{f}(\tilde{s}) = f(s)$. It is then easy to see that $\tilde{f}$ is uniformly continuous and hence extends to a uniformly continuous function, say $\hat{f}$, on S^a . The extended function $\hat{f}$ satisfies $\tilde{\rho}(\hat{f})(s) = \hat{f} \circ \rho(s) = \hat{f}(\tilde{s}) = \tilde{f}(\tilde{s}) = f(s)$. That is, $\tilde{\rho}(\hat{f}) = f$. Thus $\tilde{\rho}[C(S^a)] = AP(S)$.

Remark 1. The space obtained above coincides with the almost periodic compactification of S in [4] and [8]. This follows from [4, theorem 6.3] and the above theorem.

Remark 2. When S has sufficiently many semicharacters to separate the points of S then $\tilde{s} = \{s\}$ for each s and hence $\tilde{S}$ defined above coincides with S .

DEFINITION 1.7

Let S be a topological semigroup and X , a Banach space. For an operator $T: X \rightarrow C(S)$ and $s \in S$, define $T_s: X \rightarrow C(S)$ by $T_s(x) = (Tx)_s$, the right translate of Tx .

DEFINITION 1.8

Let $\mathcal{B}$ denote the field of Borel sets in a topological space S . If μ is a function on $\mathcal{B}$ with values in a Banach space, then the semi-variation of μ over $E \in \mathcal{B}$ is defined as

$$\|\mu\|(E) = \sup \left\| \sum_{i=1}^n \alpha_i \mu(E_i) \right\|,$$

where the supremum is taken over all finite collections of disjoint Borel sets in E and all finite set of scalars $\alpha_1, \dots, \alpha_n$ with $|\alpha_i| \leq 1$.

The following theorems are from [5]:

Theorem 1.9. Let X be a Banach space and X^* , the topological dual of X . If J is the natural embedding of X into X^{**} , the JX is X^* -dense in X^{**} .

Theorem 1.10. Let S be a compact Hausdorff space and let T be a bounded linear operator from a Banach space X into $C(S)$. Then there exists a mapping $\tau: S \rightarrow X^*$ which is continuous with the X -topology in X^* such that

$$(1) \quad Tx(s) = \tau(s)(x), \quad x \in X, \quad s \in S$$

$$(2) \quad \|T\| = \sup_{s \in S} \|\tau(s)\|.$$

Conversely, if such a map τ is given, then the operator T defined in (1) is a bounded linear operator from X into $C(S)$ with norm given by (2). The operator T is weakly compact if and only if τ is continuous with the X^{**} topology in X^* . The operator T is compact if and only if τ is continuous with the norm topology in X^* .

Theorem 1.11. Let S be a compact Hausdorff space and let T be a weakly compact operator from $C(S)$ to a Banach space X . Then there exists a vector measure μ defined on the Borel sets in S and having values in X such that

$$(a) \quad x^* \mu \text{ is in } \text{rca}(S), \quad x^* \in X^*.$$

$$(b) \quad Tf = \int_S f(s) \mu(ds), \quad f \in C(S).$$

$$(c) \quad \|T\| = \|\mu\|(S)$$

$$(d) \quad T^* x^* = x^* \mu.$$

Conversely if μ is a vector measure on the Borel sets in S to the Banach space X which satisfies (a) then the operator T defined by (b) is a weakly compact operator from $C(S)$ to X whose norm is given by (c) and whose adjoint is given by (d).

Theorem 1.12. An operator $T: C(S) \rightarrow X$ is compact if and only if the following conditions are satisfied:

subset of X .

Throughout the paper all topological spaces are assumed to be Hausdorff. Hence we shall not mention this in further discussion.

3. Operators in $BL(X, C(S))$

Let S be a compact uniform topological semigroup such that the uniformity is left translation invariant. Let H be a dense subsemigroup of S . We denote the relative uniformity on H by $\mathcal{U}$. In this section we obtain characterizations of compact and weakly compact operators from a Banach space X into $C(S)$. We crucially use theorem 1.10 and the fact that the uniformity on S is left translation invariant.

The following theorem gives the necessary and sufficient condition for an operator T in $BL(X, C(S))$ to be compact.

Theorem 2.1. *A bounded linear operator $T: X \rightarrow C(S)$ is compact if and only if the map $\theta: (H, \mathcal{U}) \rightarrow BL(X, C(S))$ defined by $\theta(s) = T_s$ is uniformly continuous in the norm topology of $BL(X, C(S))$.*

Proof. From theorem 1.10 and the fact that $(H, \mathcal{U})$ is dense in S we see that T is compact if and only if $\tau/H: (H, \mathcal{U}) \rightarrow (X^*, \|\cdot\|)$ defined by $\tau(s)(x) = Tx(s)$ is uniformly continuous. To prove the theorem first assume that T is compact. Let $\varepsilon > 0$. Then by the uniform continuity of τ , there exists a basic entourage U in $\mathcal{U}$ such that $(s, t) \in U$ implies $\|\tau(s) - \tau(t)\| < \varepsilon/2$. Since the uniformity $\mathcal{U}$ is left translation invariant, $(s, t) \in U$ implies $(us, ut) \in U$ for all $u \in S$. Hence whenever

$$(s, t) \in U, \|\tau(us) - \tau(ut)\| < \varepsilon, \quad \text{for all } u \in S. \quad (1)$$

But as

$$\begin{aligned} \|\tau_s - \tau_t\| &= \sup_{u \in S} \|\tau(us) - \tau(ut)\| \\ &= \sup_{u \in S} \sup_{\|x\| \leq 1} |\tau(us)(x) - \tau(ut)(x)| \\ &= \sup_{u \in S} \sup_{\|x\| \leq 1} |Tx(us) - Tx(ut)| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |(Tx)_s(u) - (Tx)_t(u)| \\ &= \sup_{\|x\| \leq 1} \|T_s(x) - T_t(x)\| \\ &= \|T_s - T_t\| \\ &= \|\theta(s) - \theta(t)\|, \end{aligned}$$

the uniform continuity of θ follows from (1). By reversing the chain of equalities and arguments used above we see that if θ is uniformly continuous in the norm topology of $BL(X, C(S))$ then $\tau: (H, \mathcal{U}) \rightarrow (X^*, \|\cdot\|)$ is uniformly continuous. This proves the compactness of T .

Remark 1. If we define the left translate ${}_sT$ of T in $\text{BL}(X, C(S))$ by ${}_sT(x) = {}_s(Tx)$, the left translate of Tx , then the result similar to the above theorem can be obtained. Combining the two we get the following result: If T in $\text{BL}(X, C(S))$ is compact then the map $s \rightarrow T_s$ is norm continuous if and only if the map $s \rightarrow {}_sT$ is norm continuous.

Remark 2. Let X, Y be Banach spaces on which a group G acts. Then the notion of uniformly almost periodic multiplier from X into Y is defined by [3]. If $Y = S(G)$ symmetric segal algebra on G then a characterization of a compact multipliers is obtained in [9, theorem 3]. If X and $C(S)$ are as in the above theorem we define uniformly almost periodic operators from X into $C(S)$ in a manner analogous to that of [3]. (We say that $T \in \text{BL}(X, C(S))$ is uniformly almost periodic if the set $\{T_s; s \in S\}$ is relatively compact in the uniform operator topology on $\text{BL}(X, C(S))$). Theorem 1 then shows that if a bounded linear operator $T: X \rightarrow C(S)$ is compact then it is uniformly almost periodic in this sense.

We now characterize weakly compact operators from X into $C(S)$. For Banach spaces X, Y we first define a topology on $\text{BL}(X, Y)$ as follows: For $\varepsilon > 0$ and $x^{**} \in X^*$ define a neighbourhood of zero,

$$N(\varepsilon, x^{**}) = \{T \in \text{BL}(X, Y) : \|T^{**}(x^{**})\| < \varepsilon\}.$$

The family of such sets forms a subbase for a neighbourhood system of zero in $\text{BL}(X, Y)$. We denote by τ the topology generated in this manner. One can easily check that τ is a locally convex Hausdorff topology and lies between the uniform and strong operator topology of $\text{BL}(X, Y)$.

Before proving the next theorem, we note the following: Let A be a dense subspace of a uniform space X and (Y, w) , a Banach space Y with its weak topology. Then every uniformly continuous function from A into (Y, w) has a continuous extension to X . If (Y, w) is complete there is nothing to prove. If it is not, let $\bar{Y}$ denote the completion. The result then follows from the fact that $(\bar{Y})^* = (Y, w)^*$.

Theorem 2.2. An operator T in $\text{BL}(X, C(S))$ is weakly compact if and only if the map θ defined in Theorem 2.1 is uniformly continuous with respect to the topology τ on $\text{BL}(X, C(S))$.

Proof. Since $(H, \mathcal{U})$ is dense in S , it follows from theorem 1.10 and the discussion above that T is weakly compact if and only if $\tau/H: (H, \mathcal{U}) \rightarrow (X^*, w)$ is uniformly continuous. Now assume that T is weakly compact. To prove the continuity of θ let $N(\varepsilon, x^{**})$ be any arbitrary subbasic neighbourhood of zero in τ . Then by the uniform continuity of τ , there exists a subbasic entourage U in $\mathcal{U}$ such that $(s, t) \in U$ implies $\|x^{**}(\tau(s) - \tau(t))\| < \varepsilon/8$. If J denotes the natural embedding of X into X^{**} then, by theorem 1.9, there exists a net $\{x_\alpha\}$ in X such that $J(x_\alpha)$ converges to x^{**} in the weak star topology of X^{**} . Therefore there exists $\alpha_0 = \alpha_0(s, t)$ such that

Now as $\mathcal{U}$ is translation invariant $(s, t) \in U$ implies $(us, ut) \in U$ for all $u \in S$. Hence from (1) whenever $(s, t) \in U$ and $\alpha \geq \alpha_0$ we have $|J(x_\alpha)(\tau(us) - \tau(ut))| < \varepsilon/4$, for all u in S . But

$$\begin{aligned} \sup_{u \in S} |J(x_\alpha)(\tau(us) - \tau(ut))| &= \sup_{u \in S} |\tau(us)(x_\alpha) - \tau(ut)(x_\alpha)| \\ &= \sup_{u \in S} |Tx_\alpha(us) - Tx_\alpha(ut)| \\ &= \sup_{u \in S} |(Tx_\alpha)_s(u) - (Tx_\alpha)_t(u)| \\ &= \sup_{u \in S} |T_s x_\alpha(u) - T_t x_\alpha(u)| \\ &= \|T_s(x_\alpha) - T_t(x_\alpha)\|. \end{aligned}$$

Thus for all $\alpha \geq \alpha_0$,

$$(s, t) \in U \Rightarrow \|T_s(x_\alpha) - T_t(x_\alpha)\| \leq \varepsilon/4. \quad (2)$$

Now let $f^* \in C(S)^*$ be such that $\|f^*\| \leq 1$. Again since $J(x_\alpha)$ converges to x^{**} in the weak star topology, there exists $\beta_0 = \beta_0(f^*, s, t)$ such that

$$|J(x_\alpha)(T_s^* f^* - T_t^* f^*) - x^{**}(T_s^* f^* - T_t^* f^*)| < \varepsilon/4 \text{ for all } \alpha \geq \beta_0.$$

Let $\gamma_0 \geq \alpha_0, \beta_0$. Then for $(s, t) \in U$ and $\alpha \geq \gamma_0$ we have from (2)

$$\begin{aligned} |((T_s)^{**} - (T_t)^{**})(x^{**})(f^*)| &= |x^{**}(T_s^* f^* - T_t^* f^*)| \\ &\leq |x^{**}(T_s^* f^* - T_t^* f^*) - J(x_\alpha)(T_s^* f^* - T_t^* f^*)| \\ &\quad + |J(x_\alpha)(T_s^* f^* - T_t^* f^*)| \\ &< \varepsilon/4 + |T_s^* f^*(x_\alpha) - T_t^* f^*(x_\alpha)| \\ &< \varepsilon/4 + |f^*(T_s(x_\alpha) - T_t(x_\alpha))| \\ &\leq \varepsilon/4 + \|f^*\| \|T_s(x_\alpha) - T_t(x_\alpha)\| \\ &\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

Thus if $(s, t) \in U$,

$$\|T_s^{**} x^{**} - T_t^{**} x^{**}\| = \sup_{\|f^*\| \leq 1} |T_s^{**} x^{**}(f^*) - T_t^{**} x^{**}(f^*)| < \varepsilon.$$

Equivalently, $T_s - T_t \in N(\varepsilon, x^{**})$. This proves that the map θ is uniformly continuous with respect to the topology τ on $BL(X, C(S))$. Conversely, suppose that θ is uniformly continuous in τ . To prove T is weakly compact, it is enough to show that $\tau: (H, \mathcal{U}) \rightarrow X^*$ is uniformly continuous in the weak topology of X^* . Let $V(\varepsilon, x^{**})$ be any arbitrary weak neighbourhood of zero in X^* . Since θ is uniformly continuous, for the neighbourhood of zero in τ determined by ε and x^{**} there exists U in $\mathcal{U}$ such that

$$(s, t) \in U \Rightarrow |(T_s^{**} - T_t^{**})(x^{**})(f^*)| < \varepsilon \text{ for all } \|f^*\| \leq 1.$$

Now, let $\{x_\alpha\}$ be a net in X such that $J(x_\alpha)$ converges to x^{**} in the weak topology. Then

$$\begin{aligned}
 |(T_s^{**} - T_t^{**})(x^{**})(m_e)| &= |x^{**}(T_s^* m_e - T_t^* m_e)| \\
 &= \lim_\alpha |J(x_\alpha)(T_s^* m_e - T_t^* m_e)| \\
 &= \lim_\alpha |T_s^* m_e(x_\alpha) - T_t^* m_e(x_\alpha)| \\
 &= \lim_\alpha |m_e(T_s(x_\alpha)) - m_e(T_t(x_\alpha))| \\
 &= \lim_\alpha |Tx_\alpha(s) - Tx_\alpha(t)| \\
 &= \lim_\alpha |(\tau(s) - \tau(t))(x_\alpha)| \\
 &= \lim_\alpha |J(x_\alpha)(\tau(s) - \tau(t))| \\
 &= |x^{**}(\tau(s) - \tau(t))|.
 \end{aligned}$$

Thus from (3), $(s, t) \in U$ implies $|x^{**}(\tau(s) - \tau(t))| < \varepsilon$. Equivalently, $\tau(s) - \tau(t) \in V(\varepsilon)$, which proves that the map τ is uniformly continuous in the weak topology of X . This completes the proof.

4. Operators in $BL(X, AP(S))$

Let S be a topological semigroup and X , a Banach space. As mentioned in introduction, here we apply results of §3 to the operators from X in $C(S^a)$, where S^a is the almost periodic compactification of S . (theorem 1.5). We note that the space $(\tilde{S}, \tilde{\mathcal{U}})$ constructed in §2 is dense in the compact Hausdorff topological semigroup S^a . Also the uniformity $\tilde{\mathcal{U}}$ is left translation invariant. This meets the requirements in §3.

For the simplicity of notation throughout this section we write $(S, \mathcal{U})$ for the space $(\tilde{S}, \tilde{\mathcal{U}})$ and denote the element $\tilde{s}$ of $\tilde{S}$ by s itself. As mentioned in Remark 2 of §2, $\tilde{s} = \{s\}$ when S has sufficiently many semicharacters to separate the points of S . In this case S is a locally compact abelian group. Let $\tilde{\rho}$ be the isomorphism of $C(S^a)$ onto $AP(S)$ (theorem 1.6). For proving the next result we observe here the following:

If $f \in AP(S)$ and $\hat{f}$ denotes the continuous extension of f to S^a then $\tilde{\rho}(\hat{f}) = f$. Further for any $s \in S$, $(\hat{f})_s = (\hat{f})_s$ and $\|(\hat{f})_s\| = \|f_s\|$. Also if $K \in BL(X, AP(S))$ then $\tilde{\rho}^{-1} \circ K \in BL(X, C(S^a))$ and $T_s = (\tilde{\rho}^{-1} \circ K)_s = \tilde{\rho}^{-1} \circ K_s$. The following theorem characterizes compact operators from X into $AP(S)$.

Theorem 3.1. *A bounded linear operator $K: X \rightarrow AP(S)$ is compact if and only if the map $\psi: (S, \mathcal{U}) \rightarrow BL(X, AP(S))$ defined by $\psi(s) = K_s$ is uniformly continuous in the topology of $BL(X, AP(S))$.*

that the map $\theta: (S, \mathcal{U}) \rightarrow \text{BL}(X, C(S^a))$ defined by $\theta(s) = T_s$ is uniformly continuous in the norm topology. Now from the above observations and the fact that $(S, \mathcal{U})$ is dense in S^a , we have, for s, t in S ,

$$\begin{aligned} \|T_s - T_t\| &= \sup_{\|x\| \leq 1} \|T_s(x) - T_t(x)\| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |(\tilde{\rho}^{-1} \circ K_s)(x)(u) - (\tilde{\rho}^{-1} \circ K_t)(x)(u)| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |(\widehat{Kx})(us) - (\widehat{Kx})(ut)| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |K_s x(u) - K_t x(u)| \\ &= \|K_s - K_t\| \\ &= \|\psi(s) - \psi(t)\|. \end{aligned}$$

From this the assertion of the theorem follows easily.

We now obtain the characterization of a weakly compact operator in $\text{BL}(X, \text{AP}(S))$ in terms of the topology on $\text{BL}(X, \text{AP}(S))$ defined in §3.

Theorem 3.2. *A bounded linear operator $K: X \rightarrow \text{AP}(S)$ is weakly compact if and only if the map ψ defined in theorem 3.1 is uniformly continuous in the topology τ on $\text{BL}(X, \text{AP}(S))$.*

Proof. As in the above theorem we identify the space $\text{AP}(S)$ with $C(S^a)$. Then $K: X \rightarrow \text{AP}(S)$ is weakly compact if and only if $T = \tilde{\rho}^{-1} \circ K: X \rightarrow C(S^a)$ is weakly compact. But by theorem 2.2, this is equivalent to the uniform continuity of the map $\theta: (S, \mathcal{U}) \rightarrow (\text{BL}(X, C(S^a)), \tau)$ given by $\theta(s) = T_s$. Let $N(\varepsilon, x^{**})$ be any arbitrary subbasic neighbourhood of zero in $(\text{BL}(X, \text{AP}(S)), \tau)$. Then by the uniform continuity of θ , for the neighbourhood of zero in $(\text{BL}(X, C(S^a)), \tau)$ determined by ε and x^{**} , there exists U in $\mathcal{U}$ such that $(s, t) \in U$ implies $\|(T_s)^{**}x^{**} - (T_t)^{**}x^{**}\| < \varepsilon$. But from the observations made before theorem 3.1 one can easily see that $\|(T_s)^{**}x^{**} - (T_t)^{**}x^{**}\| = \|K_s^{**}x^{**} - K_t^{**}x^{**}\|$. This shows that θ is uniformly continuous if and only if the map $s \rightarrow K_s$ is uniformly continuous with respect to topology τ . This proves the theorem.

5. Compact and weakly compact multipliers in $\text{BL}(\text{AP}(S))$

Let $S, \mathcal{U}$ and H be as in §3 and X a Banach space. In this section we first obtain characterizations of compact and weakly compact multipliers in $\text{BL}(C(S))$. When S is not compact, these results are then applied to the operators on $C(S^a)$, where S^a is the almost periodic compactification of S . As in §4, we identify the space $\text{AP}(S)$ with $C(S^a)$ to obtain the desired characterizations.

Let $\mathcal{B}$ denotes the σ -field of Borel sets in S and $\text{rca}(S)$, the space of regular countably additive scalar valued measures defined on $\mathcal{B}$.

Notation: For a set function $\mu: \mathcal{B} \rightarrow X^{**}$ and $x^* \in X^*$ we write $\mu(\cdot)x^*$ for the scalar valued set function defined by $(\mu(\cdot)x^*)(E) = \mu(E)x^*$, $E \in \mathcal{B}$. Let $E(S, \mathcal{B}, Y^{**}) =$

DEFINITION 4.1

For a set function $\mu: \mathcal{B} \rightarrow X$ and $s \in S$ define $\mu_s: \mathcal{B} \rightarrow X$ by $\mu_s(E) = \mu(Es)$, $E \in \mathcal{B}$.

DEFINITION 4.2

An operator $T: C(S) \rightarrow X$ is a multiplier if $(Tf)_s = T(f_s)$, $f \in C(S)$, $s \in S$.

For a linear operator $T: C(S) \rightarrow X$ and $s \in S$ we define $T_s: C(S) \rightarrow X$ by $T_s(f) = T(f_s)$, $f \in C(S)$. This may be compared with the definition of T_s given in §3. When $X = C(S)$ and T is a multiplier the two meanings for T_s coincide.

PROPOSITION 4.3

Let $T \in BL(C(S), X)$ and μ be the unique set function which represents T [5, theorem VI.7.2]. Then for any $s \in S$, $T_s \in BL(C(S), X)$ and the set function μ_s represents T_s .

Proof. It is easy to see that if $T \in BL(C(S), X)$, $T_s \in BL(C(S), X)$ for any $s \in S$. Let $E \in \mathcal{B}$ and $\phi_E \in C(S)^{**}$ be defined by $\phi_E(\lambda) = \lambda(E)$, $\lambda \in \text{rca}(S)$. Then by [5, theorem VI.7.2], the set function $\mu: \mathcal{B} \rightarrow X^{**}$ defined by $\mu(E) = T^{**}(\phi_E)$ represents the operator T and $T^*x^* = \mu(\cdot)x^* \in \text{rca}(S)$ for every $x^* \in X^*$. Further $\|T\| = \|\mu\|(S)$, the semivariation of μ . Now, for $x^* \in X^*$ and $f \in C(S)$, $(T^*x^*)_s(f) = T^*x^*(f_s) = x^*(T(f_s)) = x^*(T_s(f)) = (T_s)^*x^*(f)$. Thus $(T^*x^*)_s = (T_s)^*x^*$ for every $x^* \in X^*$. But then $\mu_s(E)x^* = \mu(Es)x^* = T^{**}(\phi_{Es})(x^*) = \phi_{Es}(T^*x^*) = T^*x^*(Es) = (T^*x^*)_s(E) = T_s^*x^*(E) = \phi_E(T_s^*x^*) = T_s^{**}(\phi_E)(x^*)$. Hence $\mu_s(E) = T_s^{**}(\phi_E)$ and for every $x^* \in X^*$, $\mu_s(\cdot)x^* = T_s^*x^* \in \text{rca}(S)$. As in [5, theorem VI.7.2] one can then prove that $\|T_s\| = \|\mu_s\|(S)$. This shows that the set function μ_s represents the operator T_s .

To obtain the results of this section we define topologies on $F(S, \mathcal{B}, X^{**})$ as follows:

(1) τ_ν -topology on $F(S, \mathcal{B}, X^{**})$: For $\mu \in F(S, \mathcal{B}, X^{**})$ define $p(\mu) = \|\mu\|(S)$, where $\|\mu\|(S)$ denotes the semivariation of μ . Let $\varepsilon > 0$ and $V(p, \varepsilon) = \{\mu: p(\mu) < \varepsilon\}$. Then the family of sets of this type forms a subbase at zero for a locally convex topology on $F(S, \mathcal{B}, X^{**})$. We denote this topology by τ_ν .

(2) τ' -topology on $F(S, \mathcal{B}, X^{**})$: For $\varepsilon > 0$ and $f^{**} \in C(S)^{**}$ define

$$V(\varepsilon, f^{**}) = \left\{ \mu \in F(S, \mathcal{B}, X^{**}): \sup_{\|x^*\| \leq 1} |f^{**}(\mu(\cdot)x^*)| < \varepsilon \right\}.$$

The family of sets of this type forms a subbase at zero for a topology on $F(S, \mathcal{B}, X^{**})$. We denote this topology by τ' .

We use results of §3 to characterize compact and weakly compact multipliers in $BL(C(S))$ in terms of the topologies defined in (1) and (2). It may be noted that the uniformity $\mathcal{U}$ on S is assumed to be left translation invariant.

Theorem 4.4. A multiplier T in $BL(C(S))$ is compact if and only if the map $\phi: H \rightarrow (F(S, \mathcal{B}, C(S)^{**}), \tau_\nu)$ defined by $\phi(s) = \mu_s$ is uniformly continuous, where μ_s is the unique set function which represents T_s . In this case in fact the set function μ_s is a vector measure on $\mathcal{B}$ with values in $C(S)$ and its range is relatively compact.

Proof. Since T is a multiplier, $T(f) = T(f_s) = (Tf)_s$ for every $s \in S$ and $f \in C(S)$. By

from H into $BL(C(S))$. But since the set function μ_s represents the operator T_s , we have $\|T_s - T_t\| = \|\mu_s - \mu_t\|$ (S). This shows that the map $s \rightarrow T_s$ is uniformly norm continuous if and only if $\phi: s \rightarrow \mu_s$ is uniformly continuous with respect to the topology τ_ν . The fact that μ_s is a vector measure on $\mathcal{B}$ with values in $C(S)$ and its range is relatively compact is a consequence of theorem 1.12.

Theorem 4.5. *A multiplier T in $BL(C(S))$ is weakly compact if and only if the map defined in theorem 4.4 is uniformly continuous with respect to the topology τ' on $F(S, \mathcal{B}, C(S)^{**})$ and in this case the set function μ_s is a vector measure on $\mathcal{B}$ with values in $C(S)$.*

Proof. By theorem 2.2, T is weakly compact if and only if the map $s \rightarrow T_s$ is uniformly continuous from H into $BL(C(S))$ with respect to the topology τ (defined in §2) on $BL(C(S))$, where $T_s(f) = (Tf)_s$. But since T is a multiplier, $T_s(f) = (Tf)_s = T(f_s)$. Also if μ_s is the set function which represents the operator T_s , then for every $f^* \in C(S)^*$, $T_s^* f^* = \mu_s(\cdot) f^* \in \text{rca}(S)$. Let $N(\varepsilon, f^{**})$ and $V(\varepsilon, f^{**})$ be the neighbourhoods of zero in $(BL(C(S)), \tau)$ and $(F(S, \mathcal{B}, C(S)^{**}), \tau')$ respectively. Then

$$\begin{aligned} \|(T_s)^{**} f^{**} - (T_t)^{**} f^{**}\| &= \sup_{\|f^*\| \leq 1} |f^{**}(T_s^* f^* - T_t^* f^*)| \\ &= \sup_{\|f^*\| \leq 1} |f^{**}(\mu_s(\cdot) f^* - \mu_t(\cdot) f^*)|. \end{aligned}$$

This shows that $T_s - T_t \in N(\varepsilon, f^{**})$ if and only if $\mu_s - \mu_t \in V(\varepsilon, f^{**})$. Hence the map $s \rightarrow T_s$ is uniformly τ -continuous if and only if $\phi: s \rightarrow \mu_s$ is uniformly τ' -continuous. By theorem 1.11 it follows that μ_s is a vector measure on $\mathcal{B}$ with values in $C(S)$. This proves the theorem.

Let S be a topological semigroup. To characterize compact and weakly compact multipliers on $AP(S)$ we now apply results obtained above to the operators on $C(S^a)$, where S^a denotes the almost periodic compactification of S . Let $\mathcal{B}$ denote the σ -field of Borel sets of S^a and let the space $(S, \mathcal{U})$ be as in §4. Since the space $AP(S)$ is isometrically isomorphic to $C(S^a)$, the first and the second dual of $AP(S)$ are identified with those of $C(S^a)$ up to an isometric isomorphism.

The following theorems give the characterizations of compact and weakly compact multipliers on $AP(S)$.

Theorem 4.6. *A multiplier K in $BL(AP(S))$ is compact if and only if the map $s \rightarrow \mu_s$ from $(S, \mathcal{U})$ into $(F(S^a, \mathcal{B}, AP(S)^{**}), \tau_\nu)$ is uniformly continuous, where μ_s is the unique set function which represents the operator on $C(S^a)$ corresponding to K .*

Proof. It is easy to see that a multiplier K in $BL(AP(S))$ is compact if and only if the corresponding operator on $C(S^a)$ is compact. But since $(S, \mathcal{U})$ is dense in S^a , by theorem 4.4, this is equivalent to the uniform continuity of the map $s \rightarrow \mu_s$ from $(S, \mathcal{U})$ into $F(S^a, \mathcal{B}, C(S^a)^{**})$ with respect to the topology τ_ν , where the set function μ_s represents the operator on $C(S^a)$ corresponding to K . The theorem now follows from the fact that the space $C(S^a)^{**}$ is isometrically isomorphic to $AP(S)^{**}$.

Theorem 4.7. *A multiplier K in $BL(AP(S))$ is weakly compact if and only if the map*

Proof. As in theorem 4.6 we identify the space $AP(S)$ with $C(S^a)$. The required assertion then follows by applying theorem 4.5 and using the arguments similar to those in the proof of the above theorem.

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Verschiebung and Frobenius operators*

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Abstract. Metropolis and Rota introduced the concept of the necklace ring $Nr(A)$ of a commutative ring A . When A contains $\mathbb{Q}$ as a subring there is a natural bijection $\gamma: Nr(A) \rightarrow 1 + tA[[t]]$. Grothendieck has introduced a ring structure on $1 + tA[[t]]$ while studying K -theoretic Chern classes. $Nr(A)$ comes equipped with two families of operators F_r, V_r called the Frobenius and Verschiebung operators. Mathematicians studying formal group laws have introduced two families of operators F_r and V_r on $1 + tA[[t]]$. Metropolis and Rota have not however tried to show that γ preserves these operators. They transport the operators from $Nr(A)$ to $1 + tA[[t]]$ using γ . In our present paper we show that γ does preserve all these operators.

Keywords. Necklace ring; unital power series ring; ring of Witt vectors.

Introduction

In [5] Metropolis and Rota introduce the concept of the necklace ring $Nr(A)$ of a commutative ring A . It turns out that $Nr(A)$ is a commutative ring provided with two families of operators $V_r: Nr(A) \rightarrow Nr(A)$ and $F_r: Nr(A) \rightarrow Nr(A)$ defined for every integer $r \geq 1$, referred to as the Verschiebung and the Frobenius operators respectively. Let $1 + tA[[t]]$ denote the set of unital formal power series over A in the indeterminate t . When A contains $\mathbb{Q}$ (the field of rational numbers) as a subring, the map $\gamma: Nr(A) \rightarrow 1 + tA[[t]]$ defined by $\gamma((c_1, c_2, c_3, c_4, \dots)) = \prod_{k \geq 1} (1 - t^k)^{c_k}$ can be easily seen to be a bijection. Using the bijection γ they transport the ring structure of $Nr(A)$ and the operators V_r and F_r to $1 + tA[[t]]$. It is clear that addition in $Nr(A)$ gets transformed under γ to the usual multiplication of formal power series. They write $\oplus$ for the usual multiplication of power series and refer to it as the cyclic addition. The operation $*$ in $1 + tA[[t]]$ which corresponds to the multiplication in $Nr(A)$ under γ is referred to as the cyclic multiplication. In general to be able to define the map γ we have to assume that A contains $\mathbb{Q}$ as a subring. However, when $A = \mathbb{Z}$, the ring of integers, γ as defined above turns out to be meaningful and yields a bijection $Nr(\mathbb{Z}) \rightarrow 1 + t\mathbb{Z}[[t]]$. Thus cyclic addition and cyclic multiplication can be defined in $1 + t\mathbb{Z}[[t]]$ in exactly the same way.

While studying K -theoretic Chern classes, Grothendieck introduced a ring structure on $1 + tA[[t]]$ for any commutative ring A . We will recall this definition in §1 of the

ring of the infinite cyclic group formed using additive inverses, proving using many far reaching results that the ring structure on $1 + t \mathbb{Z}[[t]]$ under cyclic addition and cyclic multiplication agrees with the ring structure defined by Grothendieck. In our present paper we will prove that a similar result is also true for all commutative rings A containing $\mathbb{Q}$ as a subring. Our proof will not involve the theory of Burnside rings. Our proof is also valid when $A = \mathbb{Z}$.

People studying formal groups have introduced two families of operators V_r : $1 + tA[[t]] \rightarrow 1 + tA[[t]]$ and F_r : $1 + tA[[t]] \rightarrow 1 + tA[[t]]$ for each integer $r \geq 1$, using elementary symmetric functions. We will recall the definition of these operators in §1 of the present paper. In our present paper we will show that when either $A = \mathbb{Z}$ or A contains $\mathbb{Q}$ as a subring, the operations V_r and F_r , defined by Metropolis and Rota on $1 + tA[[t]]$ by transporting the same operations from $\text{Nr}(A)$ by using the bijection γ , agree with the corresponding operations on $1 + tA[[t]]$, defined by people working on formal groups, using elementary symmetric functions.

In [7] the concept of the aperiodic ring $\text{Ap}(A)$ for any commutative ring A was introduced and it was shown that it was a better behaved functor than $\text{Nr}(A)$. In [8] we define operators V_r and F_r on $1 + tA[[t]]$ for each integer $r \geq 1$ by a method quite different from the method used by people studying formal groups. From the results in the present paper, it will follow that the operators V_r and F_r , defined by us on $1 + tA[[t]]$ in [8] are the same as the operators V_r and F_r , defined by people working on formal groups. In [8] we introduced a natural map $\chi: W(A) \rightarrow \text{Ap}(A)$ and using our definition of V_r and F_r on $1 + tA[[t]]$ we showed that χ is a ring homomorphism of $W(A)$ into $\text{Ap}(A)$ preserving the Verschiebung and Frobenius operators. We also characterised the image of χ . Further, we showed that $\text{Ker } \chi = 0$ if and only if the additive group of A is torsion-free. Here $W(A)$ denotes the ring of Witt vectors over A .

Throughout this paper A will denote a commutative ring with identity $1_A \neq 0$. All ring homomorphisms should preserve identity elements. For a subring B of A we want $1_B = 1_A$. We will adopt the notation in [4], [5], [6] and [7]. The Ghost ring will be defined as in [8].

1. The ring structure and the operators V_r and F_r on $1 + tA[[t]]$.

In this section we briefly recall the ring structure defined by Grothendieck on $1 + tA[[t]]$ and also the definition of the operators V_r and F_r . A good source where these are explicitly defined is [4]. Please refer to pages 119–123 of [4].

Let $\xi_1, \dots, \xi_r; \eta_1, \dots, \eta_s$ be indeterminates. We define X_i and Y_i by the equations

$$\prod_i (1 - \xi_i t) = 1 + X_1 t + X_2 t^2 + \dots \quad (1)$$

$$\prod_i (1 - \eta_i t) = 1 + Y_1 t + Y_2 t^2 + \dots \quad (2)$$

In other words the X_i and Y_i are the elementary symmetric functions in the ξ_j and η_j up to sign. Consider the products

$$\prod_{i,j} (1 - \xi_i \eta_j t) = 1 + P_1 t + P_2 t^2 + \dots \quad (3)$$

$$\prod_i (1 - \xi_i^n t) = 1 + Q_{n,1} t + Q_{n,2} t^2 + \dots \quad (4)$$

$P_i(X_1, \dots, X_i; Y_1, \dots, Y_i)$ in $X_1, \dots, X_i, Y_1, \dots, Y_i$ and $Q_{n,i}$ can be written as a polynomial in $X_1, \dots, X_{ni}$, in every case with integer coefficients. Moreover these polynomials $P_i(X; Y)$ and $Q_{n,i}(X)$ are independent of r and s (the number of ξ 's and η 's) provided $r \geq i$ and $s \geq i$ for $P_i(X; Y)$ and $r \geq ni$ for $Q_{n,i}$. This means formally we can write

$$\prod_{i=1}^{\infty} (1 - \xi_i t) = 1 + X_1 t + X_2 t^2 + \dots \quad (5)$$

and

$$\prod_{i=1}^{\infty} (1 - \eta_i t) = 1 + Y_1 t + Y_2 t^2 + \dots \quad (6)$$

imply

$$\prod_{i,j=1}^{\infty} (1 - \xi_i \eta_j t) = 1 + P_1(X_1; Y_1) t + P_2(X_1, X_2; Y_1, Y_2) t^2 + \dots \quad (7)$$

and

$$\prod_{i=1}^{\infty} (1 - \xi_i^n t) = 1 + Q_{n,1}(X_1, X_2, \dots, X_n) t + Q_{n,2}(X_1, \dots, X_{2n}) t^2 + \dots \quad (8)$$

Addition $\oplus$ in $1 + tA[[t]]$ is defined to be the same as the usual multiplication of formal power series. The product operation $\circ$ is defined as follows: Let

$$f(t) = 1 + a_1 t + a_2 t^2 + \dots$$

and

$$g(t) = 1 + b_1 t + b_2 t^2 + \dots$$

Then

$$f(t) \circ g(t) = 1 + P_1(a_1; b_1) t + P_2(a_1, a_2; b_1, b_2) t^2 + \dots \quad (9)$$

$1 + tA[[t]]$ is a commutative ring under $\oplus$ as addition and $\circ$ as multiplication. 1 is the neutral element under $\oplus$ and $1 - t$ is the identity element for the product operation $\circ$. This ring structure was introduced by Grothendieck in [3] while studying K -theoretic Chern classes.

For each integer $r \geq 1$, the operators $V_r: 1 + tA[[t]] \rightarrow 1 + tA[[t]]$ and $F_r: 1 + tA[[t]] \rightarrow 1 + tA[[t]]$ are defined by

$$V_r(f(t)) = 1 + a_1 t^r + a_2 t^{2r} + a_3 t^{3r} + \dots \quad (10)$$

$$F_r(f(t)) = 1 + Q_{r,1}(a_1, \dots, a_r) t + Q_{r,2}(a_1, \dots, a_{2r}) t^2 + \dots \quad (11)$$

V_r are "additive" homomorphisms and F_r are ring homomorphisms.

2. The isomorphism $E: W(A) \rightarrow 1 + tA[[t]]$

Let $W(A)$ denote the ring of Witt vectors over A (see pages 115–117 of [4] for the definition of $W(A)$). For any $\mathbf{a} = (a_1, a_2, a_3, \dots)$ let $E(\mathbf{a}) = \prod_{k \geq 1} (1 - a_k t^k)$. It is well known that E is a bijection. A result of Cartier [1] asserts that $E: W(A) \rightarrow 1 + tA[[t]]$ is a ring isomorphism. A proof of this result can also be found on page 121 of [4]. Using the isomorphism E^{-1} we transport the operators V_r and F_r to $W(A)$.

Before proceeding further, we will give an alternative characterization of F_r on $1 + tA[[t]]$, which takes into account the bijective nature of E .

Theorem 2.1. For any two integers $r \geq 1$ and $k \geq 1$ and any $a \in A$ we have $F_r(1/(1-at^k)) = (1/(1-a^{(r/(r,k))}t^{(k/(r,k))}))^{(r,k)}$

Proof: For the proof we need the following identities connecting the various F_r and V_r :

$$V_1 = F_1 = Id$$

$$V_r V_s = V_{rs}$$

$$F_r F_s = F_{rs}$$

$$F_r V_s = V_s F_r \text{ whenever } (r, s) = 1$$

$$F_r V_r = [r]$$

Here $[r]$ is the map $f(t) \rightarrow (f(t))^r$ of $1 + tA[[t]]$, noting that the addition $\oplus$ in $1 + tA[[t]]$ is given by the usual multiplication of formal power series. For a proof of these identities the reader can refer to 17.3.9 and 17.3.10 on page 124 of [4] and 16.2.3, 16.2.4, 16.2.6 and 16.2.9 on page 104 of [4].

Let us now evaluate $F_r(1/(1-at^k))$. We have $F_r(1/(1-at^k)) = F_r V_k(1/(1-at))$

$$= \frac{F_r}{(r, k)} F_{(r, k)} V_{(r, k)} \frac{V_k}{(r, k)} \left(\frac{1}{1-at} \right) \text{ by (13) and (14)}$$

$$= \frac{F_r}{(r, k)} \circ [(r, k)] \circ \frac{V_k}{(r, k)} \left(\frac{1}{1-at} \right) \text{ by (16)}$$

$$= [(r, k)] \frac{F_r}{(r, k)} \frac{V_k}{(r, k)} \left(\frac{1}{1-at} \right)$$

$$= [(r, k)] \frac{V_k}{(r, k)} \frac{F_r}{(r, k)} \left(\frac{1}{1-at} \right) \text{ by (15).}$$

Observe that $F_s \left(\frac{1}{1-at} \right) = \left(\frac{1}{1-a^s t} \right)$ from the very definition. Hence

$$\begin{aligned} F_r \left(\frac{1}{1-at^k} \right) &= \left\{ \frac{1}{(1-a^{(r/(r,k))} t^{(k/(r,k))})} \right\}^{(r,k)} \\ &= \left(\frac{1}{1-a^{(1/(r,k))} t^{(k/(r,k))}} \right)^{(r,k)}. \end{aligned}$$

COROLLARY 2.2.

Let $f(t) = \prod_{k \geq 1} \left(\frac{1}{1-a_k t^k} \right)$. Then for any integer $r \geq 1$ we have

$$F_r f(t) = \prod_{k \geq 1} \left(\frac{1}{1-a_k^{(r/(r,k))} t^{(k/(r,k))}} \right)^{(r,k)}.$$

Remark 2.3. In our paper [8] we defined the Frobenius operators F_r on $1 + tA[[t]]$ by formula (18). In this section we have just checked that our definition of

[8] agrees with the definition given by people working on formal groups as described in [4].

3. The map $\gamma: \text{Nr}(A) \rightarrow 1 + tA[[t]]$ for rings A containing $\mathbb{Q}$ as a subring

Let $\mathcal{C}$ denote the class of rings A satisfying the condition that $\mathbb{Q}$ is a subring of A . Throughout this section A will denote a ring belonging to $\mathcal{C}$. The following was stated in [7].

PROPOSITION 3.1.

Let $A \in \mathcal{C}$. Then every element $c \in \text{Nr}(A)$ can be written as $c = \sum_{k \geq 1} V_k \mathbf{M}(\mathbf{a}_k)$ where $\mathbf{M}(\mathbf{a}) = (M(a, 1), M(a, 2), M(a, 3), \dots) \in \text{Nr}(A)$ for any $a \in A$. Moreover the expression is unique.

As commented in [7] the proof is similar to that of theorem 2, §4 of [5].

Let $\gamma: \text{Nr}(A) \rightarrow 1 + tA[[t]]$ be the map defined by $\gamma(c) = \prod_{k \geq 1} (1/(1-t^k))^{c_k}$ where $c = (c_1, c_2, c_3, \dots) \in \text{Nr}(A)$. From the very definition of V_r we have $V_r c = x = (x_1, x_2, x_3, \dots)$ where $x_j = 0$ if $r \nmid j$ and $x_{kr} = c_k$. Hence

$$\gamma(V_r c) = \prod_{k \geq 1} \left(\frac{1}{1-t^{kr}} \right)^{c_k} = V_r \gamma(c).$$

Thus γ preserves the Verschiebung operators. The cyclotomic identity $(1/(1-at)) = \prod_{k \geq 1} (1/(1-t^k))^{M(a,k)}$ which is valid for any $a \in A$ shows that $\gamma(\sum_{k \geq 1} V_k \mathbf{M}(\mathbf{a}_k)) = \prod_{k \geq 1} (1/(1-a_k t^k))$. We have already seen that any element of $1 + tA[[t]]$ can be uniquely written as $\prod_{k \geq 1} (1/(1-a_k t^k))$. Thus γ is a bijection. Moreover $E^{-1} \circ \gamma: \text{Nr}(A) \rightarrow W(A)$ is given by $E^{-1} \circ \gamma(\sum_{k \geq 1} V_k \mathbf{M}(\mathbf{a}_k)) = \mathbf{a} = (a_1, a_2, a_3, \dots)$ in $W(A)$.

Let $\theta: \text{Nr}(A) \rightarrow \text{Gh}(A)$ be the natural map defined in [5] and [7]. Namely $\theta(c_1, c_2, c_3, \dots) = \mathbf{g} = (g_1, g_2, g_3, \dots)$ where $g_n = \sum_{d|n} dc_d$. As already seen in [7], when $A \in \mathcal{C}$, $\theta: \text{Nr}(A) \rightarrow \text{Gh}(A)$ is a ring isomorphism preserving V_r and F_r .

Theorem 3.2. Let $A \in \mathcal{C}$. Then $\gamma: \text{Nr}(A) \rightarrow 1 + tA[[t]]$ is a ring isomorphism preserving the operators V_r and F_r .

Proof. Let $\Delta: W(A) \rightarrow \text{Gh}(A)$ be the natural map defined by $\Delta(a_1, a_2, a_3, \dots) = \mathbf{g} = (g_1, g_2, \dots)$ where $g_n = \sum_{d|n} da_d^{n/d}$. We know that Δ is a ring homomorphism. We claim that

$$\begin{array}{ccc} \text{Nr}(A) & \xrightarrow{E^{-1} \circ \gamma} & W(A) \\ & \searrow \theta & \nearrow \Delta \\ & \cong & \end{array}$$

$$= \mathbf{g} = (g_1, g_2, g_3, \dots)$$

with $g_n = \sum_{d|n} da_d^{n/d}$.

Also $\theta(\sum_{k \geq 1} V_k \mathbf{M}(\mathbf{a}_k)) = \mathbf{h} = (h_1, h_2, h_3, \dots)$ where $h_n = \sum_{d|n} da_d^{n/d}$ by Proposition 6 in [7]. Hence $\theta = \Delta \circ E^{-1} \circ \gamma$.

Since $E^{-1} \circ \gamma$ and θ are bijections, it follows that Δ is a bijection. In particular $\Delta: W(A) \rightarrow \text{Gh}(A)$ is a ring isomorphism. Since θ is a ring isomorphism [Proposition 4.2 of [7]], it follows that $E^{-1} \circ \gamma: \text{Nr}(A) \rightarrow W(A)$ is a ring isomorphism. From Cartier's result [1] we know that $E: W(A) \rightarrow 1 + tA[[t]]$ is a ring isomorphism. It follows that $\gamma: \text{Nr}(A) \rightarrow 1 + tA[[t]]$ is a ring isomorphism.

We have already seen that γ preserves the operators V_r . Thus to complete the proof of theorem 3.2, we have only to prove that γ preserves the Frobenius operators.

For any $a \in A$, we have

$$\begin{aligned} F_r \gamma(V_k \mathbf{M}(\mathbf{a})) &= F_r \left(\frac{1}{1 - at^k} \right) \\ &= \left(\frac{1}{1 - a^{(r/(r,k))} t^{(k/(r,k))}} \right)^{(r,k)} \text{ by (17).} \end{aligned} \quad (18)$$

$$\text{Also } \gamma(F_r V_k \mathbf{M}(\mathbf{A})) = \gamma \left((r, k) \frac{V_k}{(r, k)} \frac{F_r}{(r, k)} \mathbf{M}(\mathbf{a}) \right) \text{ by 5.5 in [7]}$$

$$\begin{aligned} &= [(r, k)] \frac{V_k}{(r, k)} \gamma \left(\frac{F_r}{(r, k)} \mathbf{M}(\mathbf{a}) \right) \\ &= [(r, k)] \frac{V_k}{(r, k)} \gamma(\mathbf{M}(\mathbf{a}^{(r/(r,k))})) \text{ from §4 of [5]} \\ &= [(r, k)] \frac{V_k}{(r, k)} \left(\frac{1}{1 - a^{(r/(r,k))} t} \right) \\ &= \left(\frac{1}{1 - a^{(r/(r,k))} t^{(r/(r,k))}} \right)^{(r,k)} \end{aligned} \quad (19)$$

(19) and (20) yield

$$F_r \gamma(V_k \mathbf{M}(\mathbf{a})) = \gamma(F_r V_k \mathbf{M}(\mathbf{a})). \quad (20)$$

Since every element of $\text{Nr}(A)$ can be expressed as $\sum_{k \geq 1} V_k \mathbf{M}(\mathbf{a}_k)$ from (21) immediately get $F_r \gamma(c) = \gamma(F_r c)$ for any $c \in \text{Nr}(A)$. This completes the proof of theorem 3.2.

Remark 3.3. In case $A = \mathbb{Z}$ (thus $A \notin \mathcal{C}$), in diagram 1, θ is an injective ring homomorphism and so is Δ . It follows that $E^{-1} \circ \gamma: \text{Nr}(\mathbb{Z}) \rightarrow W(\mathbb{Z})$ is a ring isomorphism and hence $\gamma: \text{Nr}(\mathbb{Z}) \rightarrow 1 + t\mathbb{Z}[[t]]$ is a ring isomorphism. The proof that γ preserves V_r and F_r is exactly similar to the later part of theorem 3.2.

4. Conclusion

When A is an arbitrary commutative ring, in [7] the concept of the aperiodic ring $\text{Ap}(A)$ of A was introduced. In [8] we defined a subring $D(A)$ of $\text{Ap}(A)$ and showed that there is a natural onto ring homomorphism $\chi: W(A) \rightarrow D(A)$ preserving the operators V_r and F_r . In that paper we defined F_r by formula (18) and transported to $W(A)$ using the bijection E . Thus the present paper fills a lacuna by showing that F_r , defined by formula (18) agrees with F_r defined by people working on formal groups, by using elementary symmetric functions.

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Criterion for smoothness of Schubert varieties in $Sl(n)/B$

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Abstract. Let $G = Sl(n)$ and B , the Borel subgroup of G consisting of upper triangular matrices. Let $w \in S_n$ and $X(w) = BwB(\text{mod } B)$, the associated Schubert variety in G/B . In this paper, we give a geometric criterion for the smoothness of $X(w)$. This criterion admits a neat combinatorial description in terms of the permutation w .

Keywords. Schubert variety; smoothness, geometric criterion.

1. Introduction

In this note, we give a criterion for a Schubert variety in the flag variety $Sl(n)/B$ to be smooth.

In the series Geometry of G/P I-V, (cf. [9], [4], [7], [8], [5]), a standard monomial theory is developed for semi-simple algebraic groups. This theory consists in the construction of an explicit basis for $H^0(X, L)$, X being a Schubert variety in G/B and L being an ample line bundle on G/B . This theory has led to many important geometric and representation-theoretic consequences, one of them being the determination of the singular loci of Schubert varieties (cf [6]). In [5], using standard monomial theory, we determine explicitly the ideal defining a Schubert variety X under $X \hookrightarrow G/B$. Using this we compute in [6] the rank of the Jacobian matrix and thus obtain a description of the singularities of X . This description takes a particularly simple form for $G = Sl(n)$, which we describe below.

Let $G = Sl(n)$, $B = \{\text{upper triangular matrices in } G\}$,

$T = \{\text{diagonal matrices in } G\}$, $W = N(T)/T$ (note that $W = S_n$). For $1 \leq i < j \leq n$, let s_{ij} denote the element in W which corresponds to the transposition of i and j . For $w \in W$, let e_w denote the point wB in G/B and $X(w)$ the Schubert variety in G/B . For $\tau \leq w$, let $T(w, \tau)$ denote the Zariski tangent space to $X(w)$ at e_τ . Then we have (cf [6]),

$$\dim T(w, \tau) = \#\{s_{ij} | w \geq \tau s_{ij}\}.$$

In this note, we give a criterion for smoothness of $X(w)$, in terms of the permutation representing w . The main result of this note is the following theorem.

Theorem 1. Let $w = (a_1 \dots a_n)$. Then $X(w)$ is singular $\Leftrightarrow$ there exist i, j, k, l , $1 \leq i < j < k < l \leq n$, such that either

$$(*) \quad \begin{cases} \text{or} & 1. a_k < a_l < a_i < a_j \\ & 2. a_l < a_j < a_k < a_i. \end{cases}$$

Theorem 1 is proved in two steps. We first establish the equivalence of smoothness and equidimensionality of some canonical projections (cf. Theorem 2.1). We then show (cf. Theorem 2.2), the equivalence of the failure of equidimensionality and condition (*) of Theorem 1 above.

The sections are organised as follows: In §1, we gather some preliminaries and prove the main theorem in §2. In §3, we state a conjecture on the singular components of $X(w)$.

1. Preliminaries

Let G, B, T, W be as in the introduction. Let R be the root system of G relative to B . Let R^+ be the set of positive roots of R relative to B and let S be the set of simple roots in R^+ . We shall index the simple roots as $\alpha_1, \dots, \alpha_{n-1}$ (cf [2]). For $w \in W$, we shall denote $X(w)_r$ (resp. $X(w)_l$) the Schubert variety in G/B (resp. $B \backslash G$). If we write $X(w)$, it is to be understood that any statement on $X(w)$ holds for both $X(w)_r$ and $X(w)_l$.

PROPOSITION 1.1

(cf. [1]) $\dim X(w)_r = \dim X(W)_r = \{\alpha \in R^+ \mid w^{-1}(\alpha) < 0\}$.

PROPOSITION 1.2

(cf. [3]) Let $w \in W$. a. w has a reduced expression $w = u'_{n-1} \dots u'_1$, where u'_i is either a left-end segment of $s_i \dots s_{n-1}$ (here s_i denotes the reflection with respect to α_i) or b has a reduced expression $w = v'_1 \dots v'_{n-1}$, where v'_i is either Id or a right-end segment of $s_{n-1} \dots s_i$.

PROPOSITION 1.3

(cf [6]) For $\tau \leq w$, let $T(w, \tau)$ denote the Zariski tangent space to $X(w)$ at e_τ .

$$\dim T(w, \tau) = \#\{\alpha \in R^+ \mid w \geq \tau s_\alpha\}.$$

Notation 1.4. For $w \in W$, let $R_w = \{\alpha \in R^+ \mid w \geq s_\alpha\}$, $S_w = \{\alpha \in R_w \mid \alpha \text{ does not involve } \alpha_1\}$, $N_w = \{\alpha \in R_w \mid \alpha \text{ involves } \alpha_1\}$.

Remark 1.5. Let $w = (a_1 \dots a_n)$. Also, let $w = w' s_1 \dots s_{m-1}$, where $w' = u'_{n-1} \dots u'_2$ (cf. [3] by Proposition 1.2(a)). This in particular implies that $a_m = 1$. Let $a_1 = t$, $\#N_w = \min\{t, m\} - 1$ (here, one uses the fact that $w \geq$ the transposition s_{t-1}).

Proof. Fix j , $2 \leq j \leq m-1$. Let $i \leq j$ be such that $s_j, s_j s_{j-1} s_j, \dots, s_j s_{j-1} \dots s_{i+1} \dots s_{j-1} s_j$ are $\leq w'$, but $s_j s_{j-1} \dots s_{i+1} s_i s_{i+1} \dots s_j$ is not $\leq w'$ (note that such an i exists, since $s_1 \not\leq w'$). The facts that $s_j s_{j-1} \dots s_i \leq w'$, $s_{i-1} s_i \dots s_j \leq s_1 s_2 \dots s_{m-1}$, and $w'_1 s_1 \dots s_{m-1}$ is a reduced expression for w , imply that $s_j s_{j-1} \dots s_i s_{i-1} s_i \dots s_j \leq w$. Thus the root $\alpha = \varepsilon_{i-1} - \varepsilon_{j+1} \in R_w - R_{w'}$. From this we obtain $\# \{R_w - R_{w'}\} \geq m-1$.

Lemma 1.7. Let $w = s_{i-1} \dots s_1 w''$ (cf. Proposition 1.2(b)). Then $\# \{R_w - R_{w''}\} \geq t-1$. (Proof is similar to that of Lemma 1.6).

2. Main theorem

For $w \in W$, let $X(w)$, (resp. $X(w)_\ell$) be the Schubert variety in G/B (resp. $B \backslash G$) associated to w . Let Q_i , $1 \leq i \leq n-1$ be the parabolic subgroups defined inductively as

$$Q_1 = P_1$$

$$Q_2 = Q_1 \cap P_2$$

$$Q_3 = Q_2 \cap P_3$$

$$\vdots \quad \vdots$$

$$Q_{i+1} = Q_i \cap P_{i+1}.$$

For $0 \leq i \leq n-1$, let $\pi_{i,r}$ (resp. $\pi_{i,\ell}$) be the projection $Q_i/B \rightarrow Q_i/Q_{i+1}$ (resp. $B \backslash Q_i \rightarrow Q_{i+1} \backslash Q_i$); here, $Q_0 = G$, $Q_n = B$.

Theorem 2.1. Let $w \in W$. Then, $X(w)$ is smooth $\Leftrightarrow$ there exist $w_0, w_1, \dots, w_n, w_i \in W(Q_i)$, (with $w_0 = w$) such that for $0 \leq i \leq n-1$,

$$(\dagger) \quad \begin{cases} 1. \text{either } \pi_{i,\ell}|_{X(w)_\ell} \text{ or } \pi_{i,r}|_{X(w)_r} \text{ is equidimensional.} \\ 2. w_{i+1} \text{ is given by } X(w)_\ell \cap B \backslash Q_i \text{ or } X(w)_r \cap Q_i/B \text{ according} \\ \text{as } \pi_{i,\ell}|_{X(w)_\ell} \text{ or } \pi_{i,r}|_{X(w)_r} \text{ is equidimensional.} \end{cases}$$

Proof. a. Let $X(w)$ be smooth. We have,

$$\#N_w = \min \{t, m\} - 1 = m - 1, \text{ say}$$

(with notation as in Remark 1.5). Let T_w denote the tangent space to $X(w)$ at e_{Id} . Then the smoothness hypothesis (cf. [6], Theorem 1) implies that

$$(\#R_w =) \dim T_w = \ell(w). \quad (1)$$

Using the reduced expression $w = w' s_1 \dots s_{m-1}$, where $w' = u'_{n-1} \dots u'_2$, we have (cf. Lemma 1.6),

$$\#R_w \geq \#R_{w'} + m - 1 \geq \ell(w') + m - 1 = \ell(w).$$

Hence (1) implies equality everywhere. In particular, we have $\#R_{w'} = \ell(w')$, which implies the smoothness of $X(w')$. Let $w_1 = w'$ and w_i , $2 \leq i \leq n$, be the elements as in $(\dagger)$ (which exist by induction hypothesis applied to w'). We shall now show that $X(w)_r \rightarrow P_1 \backslash G$ is equidimensional. We have (in view of smoothness), $\#R_w = \ell(w)$. Also

$$\#R_w = \#S_w + \#N_w = \#S_w + m - 1$$

($\#N_w = m - 1$, by assumption) and

$$\ell(w) = \ell(w') + m - 1.$$

Hence we obtain $\ell(w') = \#S_w$ which in turn implies that $\#R_{w'} = \#S_w$ (by smoothness of $X(w')$). Thus we get $S_w = R_{w'}$ (note that $S_w \supseteq R_{w'}$). This now implies $w's_i < 2 \leq i \leq m - 1$ (for, $w's_i > w'$ for some i , $2 \leq i \leq m - 1$ would imply $\ell(w's_i) > \ell(w') = \#S_w \geq \#R_{w's_i} \geq \ell(w's_i)$ which is a contradiction). From this we obtain using [Lemma A.4

$$X(w)_\ell \cap B \setminus P_1 = X(w')_\ell$$

which implies the required equidimensionality.

b. Assume that (†) holds. We shall show that $X(w)$ is smooth. Let us suppose that the map $X(w)_\ell \rightarrow P_1 \setminus G$ is equidimensional and that $X(w)_\ell \cap B \setminus P_1 = X(w_1)_\ell$. As above if $w = w's_1 \dots s_{m-1}$, then we obtain (in view of equidimensionality)

$$X(w)_\ell \cap B \setminus P_1 = X(w')_\ell \text{ and } w' = w_1.$$

Now (2) implies $S_w = R_{w'}$ (note that $S_w \supseteq R_{w'}$). Also, by induction, we have, $X(w')$ smooth (note that $w' = w_1$) and hence $\#R_{w'} = \ell(w')$. Now

$$m - 1 \geq \#N_w = \#R_w - \#S_w = \#R_w - \#R_{w'} \geq m - 1$$

(cf. Lemma 1.6). Hence equality holds everywhere. In particular, we obtain $\#R_w = \#S_w + m - 1 = \#R_{w'} + m - 1 = \ell(w') + m - 1$ (since $X(w')$ is smooth) $= \ell(w)$. Thus we obtain $\dim T_w = \ell(w)$, which proves that $X(w)$ is smooth. This completes the proof of Theorem 2.1.

Theorem 2.2. Let $w \in W$. Let $w = (a_1 \dots a_n)$. Then $X(w)$ is singular $\Leftrightarrow$ there exist $i, j, 1 \leq i < j < k < l \leq n$ such that either

$$(*) \quad \begin{cases} 1. & a_k < a_\ell < a_i < a_j \\ \text{or} & \\ 2. & a_\ell < a_j < a_k < a_i \end{cases}$$

Proof. We shall show that (*) holds $\Leftrightarrow$ (†) of Theorem 2.1 does not hold, from which the required result would follow (in view of Theorem 2.1).

a. Given that (*) holds, to show that (†) does not hold. If possible, let us assume that (†) holds. Let us suppose that $X(w)_\ell \rightarrow P_1 \setminus G$ is equidimensional and that

$$(I) \quad X(w_1)_\ell = X(w)_\ell \cap B \setminus P_1.$$

Then writing $w = w's_1 \dots s_{m-1}$, we have $w' = w_1$. Hence in view of (I) we have, $w's_i < 2 \leq i \leq m - 1$. Let $w' = (1b_2b_3 \dots b_n)$. Then $w = w'(23 \dots m1 \dots) = (b_2b_3 \dots b_m1 \dots b_n)$, where

$$(A) \quad b_2 > b_3 > \dots > b_m$$

(since $w' > w's_i$; note that $w' > w's_i \Leftrightarrow w'(\varepsilon_i - \varepsilon_{i+1}) = (\varepsilon_{b_i} - \varepsilon_{b_{i+1}})$ is > 0). We may assume by induction that (*) does not hold for $(a_2a_3 \dots a_n)$. Hence for (*) to hold for $(a_1a_2 \dots a_n)$ we must have that $1 < a_1 < a_2 < a_3 < \dots < a_n$. This is a contradiction since $a_1 = 1$.

implies $a_i < a_j$ in (1) (resp. $a_j < a_k$ in (2)), where a_i, a_j belong to $(b_2, \dots, b_m)$ in (1) (resp. $a_j, a_k \in \{b_2, \dots, b_m\}$ in (2)), which is not possible in view of (A) above (note that for a_i in $\{b_2, \dots, b_m\}$, $a_i = b_{i+1}$).

b. Given that (†) of Theorem 2.1 does not hold, to show (*) holds. Using induction hypothesis, we may assume that (†) of Theorem 2.1 holds for $i \geq 2$. Let us then suppose that $\pi_{1,r}$ and $\pi_{1,r'}$ are not equidimensional. Then considering $w = w's_1 \dots s_{m-1}$ and $w = s_{r-1} \dots s_1 w''$, we have the existence of integers p, q such that

$$w = (t \dots p \dots p+1 \dots) \text{ (resp. } w = (t \dots a_q a_{q+1} \dots 1 \dots)$$

where $p+1 \leq t$ and $a_q < a_{q+1}$. We shall now show that (*) holds. Let $a_q = b$, $a_{q+1} = c$. We divide the proof into the following two cases.

Case 1. 1 comes after $p+1$.

In this case (2) holds with $a_i = t$, $a_j = p$, $a_k = p+1$, $a_\ell = 1$.

Case 2. 1 comes before $p+1$.

We first note that $p+1 \neq b, c$ (since b and c come before 1 while $p+1$ comes after 1). We further divide this case into the following three sub cases.

Sub case 2(a). $b > p+1$.

In this case, (1) holds with $a_i = b$, $a_j = c$, $a_k = 1$, $a_\ell = p+1$.

Sub case 2(b). $b < p+1$, $t > c$.

In this case (2) holds with $a_i = t$, $a_j = b$, $a_k = c$, $a_\ell = 1$.

Sub case 2(c). $b < p+1$, $t < c$.

In this case, (1) holds with $a_i = t$, $a_j = c$, $a_k = 1$, $a_\ell = p+1$. This completes the proof of Theorem 2.2.

Remark 2.3. For $G = S\ell(4)$, $X(w)$, $w = (4231), (3412)$ are the only two singular Schubert varieties. The phenomenon in the general case thus seems to be just a generalization of that in $S\ell(4)$.

3. A conjecture

In this section, we state a conjecture giving the components of the singular locus of $X(w)$. We first start with the following remark on $S\ell(4)$.

Remark 3.1 a. For $w = (3412)$, $\tau = (1324)$ is a singularity (and is in fact the only singularity).

b. For $w = (4231)$, $\tau = (2143)$ is a singularity (and is in fact the only singularity).

A similar phenomenon seems to hold in the general case, as given by the following Lemma.

Lemma 3.1. Let $X(w)$ be singular. Let $w = (a_1 \dots a_n)$. Let i, j, k, ℓ be such that $1 \leq i < j < k < \ell \leq n$.

1. Let $a_k < a_\ell < a_i < a_j$. Let τ be obtained from w by replacing a_i, a_j, a_k, a_ℓ by a_k, a_i, a_ℓ, a_j . Then e_τ is singular on $X(w)$.

2. Let $a_i < a_j < a_k < a_l$. Let τ be obtained by replacing a_i, a_j, a_k, a_l by a_j, a_l, a_i, a_k respectively. Then e_τ is singular on $X(w)$.

Proof. Let $T(w, \tau)$ be the tangent space to $X(w)$ at e_τ . Let

$$A = \{s_\alpha \mid w \geq \tau s_\alpha\} \text{ and } B = \left\{ (x, y) \left| \begin{array}{l} 1. x = a_r, y = a_s, \text{ for some } r < s \\ 2. x > y \end{array} \right. \right\}.$$

We have, $\dim T(w, \tau) = \#A$ and $\dim X(w) = \#B$. We shall prove the result by showing that $\#A > \#B$. We shall prove the result for (1) (the proof for (2) is similar). For simplicity of notation, let us denote a_i, a_j, a_k, a_l by c, d, a, b respectively, so that we have $a < b < c < d$. We have

$$w = (\dots c \dots d \dots a \dots b \dots)$$

$$\tau = (\dots a \dots c \dots b \dots d \dots)$$

(note that the entries in w and τ are the same in the rest of the places). We shall now define an injective mapping $f: B \rightarrow A$. Let $x = a_u, y = a_v$. In the discussion below (where $1 \leq p < q \leq n$) will denote the transposition $(1 \dots (p-1)p(q+1) \dots (q-1)p(q+1) \dots)$. We define f by distinguishing the following cases.

Case 1. $x, y \notin \{a, b, c, d\}$.

Set $f(x, y) = s_{uv}$ (note that τ itself is $\geq \tau s_{uv}$).

Case 2. $x \notin \{a, b, c, d\}, y \in \{a, b, c, d\}$.

We divide this case into the following four sub cases.

Case 2(a). $u < i$, i.e., x comes before c , so that $w = (\dots x \dots c \dots d \dots a \dots b \dots)$. Set $\tau = (\dots x \dots a \dots c \dots b \dots d \dots)$. Set

$$f(x, y) = \begin{cases} s_{uj}, & \text{if } y = c \\ s_{ul}, & \text{if } y = d \\ s_{ui}, & \text{if } y = a \\ s_{uk}, & \text{if } y = b \end{cases}$$

Case 2(b). $i < u < j$.

We have $w = (\dots c \dots x \dots d \dots a \dots b \dots)$, $\tau = (\dots a \dots x \dots c \dots b \dots d \dots)$. Set

$$f(x, y) = \begin{cases} s_{ul}, & \text{if } y = d \\ s_{uk}, & \text{if } y = b \\ s_{uj}, & \text{if } y = a \end{cases}$$

Case 2(c). $j < u < k$.

We have $w = (\dots c \dots d \dots x \dots a \dots b \dots)$ and $\tau = (\dots a \dots c \dots x \dots b \dots d \dots)$. Set

$$f(x, y) = \begin{cases} s_{ul}, & \text{if } y = b \\ s_{uk}, & \text{if } y = a \end{cases}$$

Case 3. $x \in \{a, b, c, d\}$, $y \notin \{a, b, c, d\}$.

We divide this case into the following sub cases.

Case 3(a): $i < v < j$.

We set $f(x, y) = s_{iv}$.

Case 3(b): $j < v < k$.

We have $w = (\dots d \dots y \dots a \dots b \dots)$, $\tau = (\dots a \dots c \dots y \dots b \dots d \dots)$. Set

$$f(x, y) = \begin{cases} s_{iv}, & \text{if } x = c \\ s_{jv}, & \text{if } x = d \end{cases}$$

Case 3(c): $k < v < \ell$.

We have $w = (\dots c \dots d \dots a \dots y \dots b \dots)$, $\tau = (\dots a \dots c \dots b \dots y \dots d \dots)$. Set

$$f(x, y) = \begin{cases} s_{iv}, & \text{if } x = a \\ s_{jv}, & \text{if } x = c \\ s_{kv}, & \text{if } x = d \end{cases}$$

Case 3(d): $\ell < v$.

We have $w = (\dots c \dots d \dots a \dots b \dots y \dots)$, $\tau = (\dots a \dots c \dots b \dots d \dots y \dots)$. Set

$$f(x, y) = \begin{cases} s_{iv}, & \text{if } x = a \\ s_{jv}, & \text{if } x = c \\ s_{kv}, & \text{if } x = b \\ s_{\ell v}, & \text{if } x = d \end{cases}$$

Case 4: $x, y \in \{a, b, c, d\}$. Set

$$f(x, y) = \begin{cases} s_{ij}, & \text{if } (x, y) = (c, a) \\ s_{jk}, & \text{if } (x, y) = (c, b) \\ s_{j\ell}, & \text{if } (x, y) = (d, a) \\ s_{k\ell}, & \text{if } (x, y) = (d, b) \end{cases}$$

Thus we obtain an injective mapping $f: B \hookrightarrow A$. Now we have at least one element in A which $\notin f(B)$, namely s_{ik} . Hence we obtain $\#A \geq \#B$. From this it follows that e_τ is singular on $X(w)$. This completes the proof of Lemma 3.1.

We conclude this note by stating a conjecture on the singular components of $X(w)$.

Conjecture. Let $w = (a_1 \dots a_n)$. Let Z be the set of all $\tau' \leq w$ such that either (1) or (2) below holds.

1. There exist i, j, k, ℓ , $1 \leq i < j < k < \ell \leq n$ such that

(a) $a_k < a_\ell < a_i < a_j$

(b) if $\tau' = (b_1 \dots b_n)$, then there exist i', j', k', ℓ' , $1 \leq i' < j' < k' < \ell' \leq n$ such that $b_{i'} = a_k$, $b_{j'} = a_i$, $b_{k'} = a_\ell$, $b_{\ell'} = a_j$

(c) if τ (resp. w') is the element obtained from w (resp. τ) by replacing a_i, a_j, a_k, a_ℓ respectively by a_k, a_i, a_ℓ, a_j (resp. $b_{i'}, b_{j'}, b_{k'}, b_{\ell'}$ respectively by $b_{j'}, b_{\ell'}, b_{i'}, b_{k'}$), then $\tau' \geq \tau$ and $w' \leq w$.

(c) if τ (resp. w') is the element obtained from w (resp. τ') by replacing a_i, a_j, a_k, a_l respectively by a_j, a_l, a_i, a_k (resp. $b_{i'}, b_{j'}, b_{k'}, b_{l'}$, respectively by $b_{k'}, b_{i'}, b_{l'}, b_{j'}$), then $\tau' \geq \tau$ and $w' \leq w$.

Conjecture: Singular locus of $X(w) = \bigcup_{\lambda} X(\lambda)$ where λ runs over the maximal (under the Brühart order) elements of Z .

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Vector fields and framings on isolated complete intersection singularities

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Abstract. A theorem is proved for deciding as to when the complex orthogonal complement of a vector field on an isolated, complete intersection germ, is a trivial vector bundle.

Keywords. Vector field; singularity link.

1. Introduction

In this note the following question is studied: Given (V, P) , a n -dimensional, isolated, complete intersection germ in some complex space $\mathbb{C}^{n+k}$, and a (holomorphic or continuous) vector field Z on V , non-singular away from P , when is the bundle Z^* , normal to the $\mathbb{C}$ -distribution on $V - \{P\}$ spanned by Z , a trivial vector bundle? I prove.

Theorem. *The bundle Z^* is topologically trivial over $\mathbb{C}$ if and only if the index of Z in V is a multiple of $(n-1)!$.*

Since V is singular at P , one has to make sense of the concept of the index of a vector field in V [3, 10]. In §2, I translate the question above into a general problem in differential topology, which is solved in §3. It is worth noting that the theorem above arises a basic question whose answer is not known, not even in the simplest case when V is non-singular at P : If Z^* is not trivial, how many sections does this bundle admit?

2.

Let $f_1, \dots, f_k$ be holomorphic functions on (an open set in) $\mathbb{C}^{n+k}$ defining the germ of V at P , so that the map

$$f = (f_1, \dots, f_k): (\mathbb{C}^{n+k}, P) \rightarrow (\mathbb{C}^k, 0)$$

$$J(Z) = (Z, \nabla f_1, \dots, \nabla f_k): M \rightarrow W$$

where M is the link of P in V , as defined in [7], and W is the Stiefel manifold of complex, orthonormal $(k+1)$ -frames in $\mathbb{C}^{n+k}$; W is $(2n-2)$ -connected, $\pi_{2n-1}(W) \cong \mathbb{Z}$, see [2], so that $J(Z)$ has a well defined degree in $\mathbb{Z}$. By definition [3, 10], the index of Z in V is the degree of $J(Z)$, denoted $\text{Ind}(Z, V)$.

Let us now choose $\varepsilon > 0$ sufficiently small [7], so that every sphere in $\mathbb{C}^{n+k}$ centered at P and of radius less or equal to ε intersects $V = f^{-1}(0)$ transversely, and let $\delta > 0$ be sufficiently small [7] with respect to ε , so that every fiber $V_t = f^{-1}(t)$ with $|t| < \delta$ intersects the sphere S_ε transversely. We let $V' = V_t \cap D_\varepsilon$, where D_ε is the closed ε -disc around P and $0 < |t| < \delta$ is fixed. Then V' is the Milnor fibre of P ; it is a parallelizable manifold with the homotopy type of a wedge of n -spheres, [6, 7]. Moreover, the boundary of V' is isotopic to $M = V \cap S_\varepsilon$, the link of P , and the vector field Z can be regarded as a continuous, nowhere-zero, vector field Z' on a neighbourhood of the boundary $\partial V' \subset V'$. As such, Z has a well defined index in V' : the total number of zeros, counted with their local indices [8], of a continuous extension of Z' to all of V' ; by classical homotopy theory (see [11]), this number $\text{Ind}(Z', V')$ does not depend on the choice of the extension of Z' , and as noted in [3] it equals $\text{Ind}(Z, V)$, because the gradient vector fields $\nabla f_1, \dots, \nabla f_k$ are everywhere linearly independent over V' , and they are normal to V' .

Since V is a cone over M , by [7], it follows that the bundle Z^* over $V - \{P\}$ is trivial if and only if it is trivial restricted to M . Also $Z^*|_M$ is isotopic to the bundle $(Z')^*$, normal to Z' in $V'|\partial V'$. Thus we have established the following lemma.

Lemma 1. The theorem above is equivalent to claiming that $(Z')^*$ is trivial if and only if $\text{Ind}(Z', V')$ is a multiple of $(n-1)!$.

This lemma will follow at once from the results in §3 below, thus proving the theorem stated in the introduction.

3.

We now let X be an arbitrary $2n$ -dimensional, compact, almost complex manifold with non-empty boundary M and with trivial tangent bundle TX , and we let Z be a nowhere-zero, continuous vector field on a neighbourhood of M in X . Let $\text{Ind}(Z, X)$ denote the total index of Z in X ; as explained before, this means $\text{Ind}(Z, X)$ is the total sum of the zeros of a continuous extension of Z to X , with finite singularities and counted with their local indices [8].

Lemma 2. If $\text{Ind}(Z, X) \equiv 0 \pmod{(n-1)!}$, then Z can be completed to a continuous $\mathbb{C}$ -trivialization of $TX|_M$.

This lemma can be proved directly using obstruction theory. Instead we give a simpler, geometric proof that uses the Hirsh-Poenaru immersion theorem and it has the advantage of proving the lemma above by providing an explicit trivialization of

$TX|M$ with Z as one of its sections, up to homotopy. (Moreover; the Chern classes of X relative to this trivialization are all zero except the top one, which equals $\text{Ind}(Z, X)$. This allows us to think of this number as being a Chern class [5]). To prove lemma 2 we note that, by the theorem of Poincaré-Hopf for closed manifolds [8], there exist around M vector fields of every index in X . Furthermore, by the theorem of Hirsh-Poenaru [9], there exists an immersion

$$\mathcal{J}: X \rightarrow \mathbb{R}^{2n}.$$

Thus, if τ is a vector field around M , then we may define a map $\psi(\tau): M \rightarrow S^{2n-1}$ by

$$\psi(\tau)(x) = \frac{\mathcal{J}_*(\tau(x))}{|\mathcal{J}_*(\tau(x))|},$$

where $\mathcal{J}_*$ is the derivative of $\mathcal{J}$, and the degree of $\psi(\tau)$ equals the index of τ in X , because τ can always be extended to all of X minus one point x° around which the immersion can be assumed to be an embedding. It then follows that two vector fields τ and τ° around M are homotopic if and only if they have the same index in X . Now, let Z be as in lemma 2 and let $\text{Ind}(Z, X) = t \cdot (n-1)!$ for an appropriate $t \in \mathbb{Z}$. We choose Z° to be a vector field around M of index t in X , and we define a map

$$\phi(Z^\circ): M \rightarrow U(n)$$

by considering $\psi(Z^\circ)$ as above and composing it with the generator of $\pi_{2n-1}(U(n))$, which is isomorphic to $\mathbb{Z}$, see [1]. If F° denotes a trivialization of TX , then we twist F° over M as in [4] using $\phi(Z^\circ)$. We claim that the new trivialization $F(Z)$ of $TX|M$ that we obtain has Z as one of its n -sections, up to homotopy. To see this we observe first that $\psi(Z^\circ)$ has degree t , by the previous discussion. Also, there is a fibration $U(n-1) \hookrightarrow U(n) \rightarrow S^{2n-1}$, and an associated long exact homotopy sequence,

$$(*) \quad \cdots \rightarrow \pi_{2n-1}(U(n)) \xrightarrow{p_*} \pi_{2n-1}(S^{2n-1}) \rightarrow \pi_{2n-2}(U(n-1)) \rightarrow \pi_{2n-2}(U(n)) \rightarrow \cdots$$

By Bott's calculations [1], p_* is multiplication by $(n-1)!$. Thus, each section of $F(Z)$ has index $t \cdot (n-1)!$ in X , and the result follows.

Lemma 3. *Let the manifold X have the homotopy type of a wedge of n -spheres. If Z is a vector field around M , non-singular and such that its complex orthogonal complement Z^* is $\mathbb{C}$ -trivial, then $\text{Ind}(Z, X) \equiv 0 \pmod{(n-1)!}$.*

Proof. This is based on a lemma of [5] which states that if F is a complex trivialization of $TX|M$, then F extends to all of X minus one point if n is odd, or to X minus an embedded torus $T = S^1 \times D^n$ if n is even. Suppose first that n is odd and let β be a trivialization of Z^* . Then Z and β define a trivialization F of $TX|M$. By the result of [5] stated above, F extends to all of X minus a point x° , thus the index of Z in X equals the degree of a complex framing F'' on a small sphere S_σ around x° , i.e. the index of one of the sections that defines F'' . The result then follows from the exact sequence (*). Now suppose n is even, let β be as above and let F be the trivialization of $TX|M$ defined by Z and β . Then F extends to $X - T$ and the result follows from

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A class of totally geodesic foliations of Lie groups

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Abstract. This paper is devoted to classifying the foliations of G with leaves of the form gKh^{-1} where G is a compact, connected and simply connected Lie group and K is a connected closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space. In the case when $G/K = S^n$, the homotopy type of space of such foliations is also given.

Keywords. Foliation; rank-1 Riemannian symmetric space; cutlocus.

1. Introduction

The study of fibrations of spheres by great spheres is a very interesting problem in geometry and it is very important in the theory of Blaschke manifolds. In [1], Gluck and Warner have studied the great circle fibrations of the three spheres. In that paper they have proved very interesting results. When we look at the problem group theoretically, we see that all the results of [1] go through, for foliations of G with leaves of the form gKh^{-1} where G is a compact, connected and simply connected Lie group and K is a connected closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space (see [2]), except perhaps the theorem 3 for the pair (G, K) such that $G/K = \mathbb{CP}^n$, $\mathbb{HP}^n$ etc. Thus this paper is a group theoretic generalisation of [1].

Throughout we will be working with foliations of G with leaves of the form gKh^{-1} where $g, h \in G$. We call such foliations *admissible*.

Let us fix some notations which we will be using throughout:

G -Compact, connected, simply connected Lie group.

K -connected and closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space.

$\mathfrak{g}$ -Lie algebra of G .

$\mathfrak{k}$ -Lie algebra of K .

$Gr(\mathfrak{g}, \mathfrak{k}) = \{Ad_g(\mathfrak{k}); g \in G\}$.

$\tilde{Gr}(\mathfrak{g}, \mathfrak{k})$ -universal cover of $Gr(\mathfrak{g}, \mathfrak{k})$.

$\mathcal{F}$ -admissible foliation of G .

Unless we specify, we will always be working with continuous foliations.

Let $\mathcal{L} := \{gKh^{-1}; g, h \in G\}$. $\mathcal{L}$ is the set of all possible leaves of the foliations $\mathcal{F}$ of G . The group $G \times G$ acts on $\mathcal{L}$ as follows:

given by

$$((g, h), L) \mapsto gLh^{-1}.$$

This action of $G \times G$ on $\mathcal{L}'$ is transitive. Let us denote the isotropy group of the action at the point K by I_K . Then we have the following

Lemma 1. $I_K = \{(g, h) \in N(K) \times N(K) : gh^{-1} \in K\}$

Proof.

$$gKh^{-1} = K \Rightarrow gh^{-1} \in K.$$

Hence

$$gKg^{-1} = gKh^{-1}hg^{-1} = Khg^{-1} = K$$

$\Rightarrow g \in N(K)$, and hence $h \in N(K)$. Therefore,

$$gKh^{-1} = K \Rightarrow g, h \in N(K).$$

Conversely, let $g, h \in N(K)$ be such that $gh^{-1} \in K$.

$$gKh^{-1} = gKg^{-1}gh^{-1} = Kgh^{-1} = K.$$

Hence $(g, h) \in I_K$. Thus

$$I_K = \{(g, h) \in N(K) \times N(K) : gh^{-1} \in K\}.$$

(Q.E.D.)

From the above lemma it follows that

$$\begin{aligned} G \times G / I_K \\ \cong \mathcal{L}'. \end{aligned}$$

We topologise $\mathcal{L}'$ using the bijection $G \times G / I_K \cong \mathcal{L}'$. From now on we shall denote by $G \times G / I_K$ for $\mathcal{L}'$.

Let $\mathcal{F}$ be an admissible foliation of G . We define an equivalence relation $\sim$ on G as follows. For $x, y \in G$, define $x \sim y$ iff $x, y \in l$ for some leaf l in $\mathcal{F}$. This is an equivalence relation on G . Let us denote by $M_{\mathcal{F}} = G/\sim$, the quotient space of the equivalence relation. Then it can easily be seen that $G \rightarrow M_{\mathcal{F}}$ is a fibre bundle with fibres gKh^{-1} . If we write down the homotopy long exact sequence of this fibration, we see that $M_{\mathcal{F}}$ is simply connected.

Furthermore, we have a continuous inclusion $i: M_{\mathcal{F}} \hookrightarrow G \times G / I_K$ which is a homeomorphism on to its image. We will identify $M_{\mathcal{F}}$ with its image. Thus given an admissible foliation $\mathcal{F}$ of G , we get a simply connected topological submanifold $M_{\mathcal{F}} \subseteq G \times G / I_K$. Observe that I_K is a closed subgroup of $G \times G$ such that $K \cdot K = I_K \cap N(K) \times N(K)$. The fact that G/K is a rank-1 Riemannian symmetric space implies that $(G/K) \times (G/K) \rightarrow G \times G / I_K$ is a finite sheeted simply connected covering. Since $M_{\mathcal{F}}$ is simply connected, we can lift $M_{\mathcal{F}}$ to a simply connected submanifold of $G \times G / I_K$. We note that there is no uniqueness in the lifting and any two liftings will differ by an element of $\pi_1(G \times G / I_K) \cong \pi_1(G \times G / I_K)$.

choosing an orientation of the great circle fibration of S^3 . Because of this reason, after fixing one lifting we say that our foliation is *oriented*.

From now on we will be working with a fixed oriented admissible foliation only. After this much of preliminaries, presently we prove two lemmas from which our main theorem of this section is almost immediate.

Lemma 2. For $g, h \in G$,

$$gKh^{-1} \cap K \neq \emptyset \Leftrightarrow d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e}),$$

where d denotes the Riemannian distance on G/K and $\bar{g}$ denotes the image of the left coset gK in G/K .

Proof.

$$gKh^{-1} \cap K \neq \emptyset \Leftrightarrow gK \cap Kh \neq \emptyset.$$

This is true iff $\bar{g} \in K\bar{h}$ where $K\bar{h}$ is the K -orbit of $\bar{h}$ in G/K , the K action being the induced action of G on G/K . G (and hence K) acts on G/K via isometries. Furthermore K fixes $\bar{e}$. This means that $d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e})$.

Conversely, let $d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e}) = r$ (say). Let $S(\bar{e}, r) \subseteq T_{\bar{e}}(G/K)$ be the geodesic sphere of radius r . Let $\gamma_{\bar{g}}$ be the geodesic from $\bar{e}$ to $\bar{g}$ (respectively, $\gamma_{\bar{h}}$ be the geodesic from $\bar{e}$ to $\bar{h}$). Both $\gamma'_{\bar{g}}(0)$ and $\gamma'_{\bar{h}}(0) \in S(\bar{e}, r)$. Since G/K is a rank-1 Riemannian symmetric space, there exists a $k \in K$ such that $k\gamma'_{\bar{g}}(0) = \gamma'_{\bar{h}}(0)$ (see [2]). This means that $\bar{g}, \bar{h}$ are in the same K -orbit in G/K .

Q.E.D.

So, given $M_{\mathcal{F}} \subseteq (G/K) \times (G/K)$, we know from the above lemma that for any two distinct points of $M_{\mathcal{F}}(\bar{g}_1, \bar{h}_1)$ and $(\bar{g}_2, \bar{h}_2)$ of $M_{\mathcal{F}}$, $d(\bar{g}_1, \bar{g}_2) \neq d(\bar{h}_1, \bar{h}_2)$. But what is more interesting is

Lemma 3. Either $d(\bar{g}_1, \bar{g}_2) > d(\bar{h}_1, \bar{h}_2)$ or $d(\bar{g}_1, \bar{g}_2) < d(\bar{h}_1, \bar{h}_2)$ for any two distinct points $(\bar{g}_1, \bar{h}_1)$ and $(\bar{g}_2, \bar{h}_2)$ in $M_{\mathcal{F}}$.

Proof. The proof follows easily from the fact that $M_{\mathcal{F}}$ is path connected.

Given an oriented admissible foliation $\mathcal{F}$ we have a submanifold $M_{\mathcal{F}} \subseteq (G/K) \times (G/K)$. We can ask if the converse is true or not. The answer to this question is our main theorem in this section which we prove now.

Theorem 1. A submanifold $M \subseteq G/K \times G/K$ corresponds to an oriented admissible foliation $\mathcal{F}$ of G iff $M = \text{Graph}(f)$, for some strictly distance decreasing map f from either of the G/K factor to the other.

Hence we may (and we will) assume that $d(\bar{g}_1, \bar{g}_2) > d(\bar{h}_1, \bar{h}_2)$. This implies that $d(\bar{g}_1, \bar{g}_2) > 0$ always. Hence the map

$$\Phi: M_{\mathcal{F}} \rightarrow \frac{G}{K}$$

defined by $\Phi(\bar{g}, \bar{h}) = \bar{g}$ is continuous and injective. This implies that Φ is an open map and hence a homeomorphism of $M_{\mathcal{F}}$ on to (G/K) .

Now we define another map

$$f: \frac{G}{K} \rightarrow \frac{G}{K}$$

by $f(\bar{g}) = \bar{h}$, where $(\bar{g}, \bar{h}) \in M_{\mathcal{F}}$. Then by definition f is strictly distance decreasing and $M_{\mathcal{F}} = \text{graph}(f)$.

Conversely, let $M = \text{graph}(f)$. We will show that $M = M_{\mathcal{F}}$ for some oriented admissible foliation $\mathcal{F}$ of G . Let us denote by $\mathcal{F}$ the collection of all submanifolds of the form gKy^{-1} where $y \in f(\bar{g})$ is any element in the coset $f(\bar{g})$, i.e., $\mathcal{F} = \{gKy^{-1} : g \in G \text{ and } y \in f(\bar{g})\}$. We will show that $\mathcal{F}$ is an oriented foliation of G . Since f is strictly distance decreasing, $g_1Ky_1^{-1} \cap g_2Ky_2^{-1} = \emptyset$ for $\bar{g}_1 \neq \bar{g}_2$ in G/K and $y_i \in f(\bar{g}_i)$. First we shall show that G is a continuous union of submanifolds L where $L \in \mathcal{F}$. Since $G \rightarrow G/K$ is a principal K -bundle, local sections always exist. We will use this fact to prove our claim.

Consider $(\bar{g}, f(\bar{g})) \in \text{Graph}(f)$. Let V be an open neighbourhood of $f(\bar{g})$ such that G has a local section S_2 over V . Let U be an open neighbourhood of $\bar{g}$ such that G has a local section S_1 over U . We may assume that U is contained in $f^{-1}(V)$ which is an open neighbourhood of $\bar{g}$. Hence $f(U) \subseteq V$. Now we define a map $\Phi_U: U \times K \rightarrow G$ by $\Phi_U(\bar{g}, k) = S_1(\bar{g})kS_2(f(\bar{g}))^{-1}$. The map Φ_U is continuous by the definition. Clearly Φ_U is injective.

This shows that Φ_U is open. Since G/K is compact, we can choose finitely many of those U 's which cover whole of G . Hence the subset of G which is the finite union of the images of Φ_U must be open, i.e. $\cup \Phi_U(U \times K) \subseteq G$ is open. Since M is compact and each gKh^{-1} is a closed submanifold of G , this subset $\cup \Phi_U(U \times K)$ must be compact and hence closed. Thus $M = \text{graph}(f)$ defines an oriented admissible foliation of G . Q.E.D.

Remark. Any constant map $f: G/K \rightarrow G/K$ corresponds to a coset foliation, and conversely.

2. Differentiability of the foliation

So far we have not said anything about the differentiability of the foliation. In this section we shall see how the differentiability of the foliation is related to that of the map.

If we start with a strictly distance decreasing differentiable map f such that $\|df\| < 1$, we can conclude that the foliation has to be differentiable. We can ask if the converse is true or not. Our main theorem in this section answers this question affirmatively.

Theorem 2. *An oriented foliation $M_{\mathcal{F}} = \text{graph}(f)$ is differentiable iff f is differentiable and $\|df\| < 1$.*

Given $M_{\mathcal{F}}$ we get a continuous integrable distribution of the tangent bundle TG of G . Consider $(\bar{g}, \bar{h}) \in M_{\mathcal{F}}$. Then gKh^{-1} is a leaf for the foliation $\mathcal{F}$ of G . For $p \in gKh^{-1}$, the tangent space at p tangential to the foliation is $L_{gkh^{-1}*}(Ad_h(\mathbf{k}))$.

For a foliation $M_{\mathcal{F}}$ the leaf gKh^{-1} , passing through the identity e of G , is of the form gKg^{-1} . For there exists a $k \in K$ such that $gkh^{-1} = e$. This shows that $h = gk$ and hence $gKh^{-1} = gKg^{-1}$. The tangent space tangential to this leaf at any point gkg^{-1} is $Ad_g(\mathbf{k})$.

Let us denote by $Gr(\mathbf{g}, \mathbf{k})$ the set of all n -dimensional subspaces of $\mathbf{g}$ which are of the form $Ad_g(\mathbf{k})$ where $n = \dim K$. i.e. $Gr(\mathbf{g}, \mathbf{k}) := \{Ad_g(\mathbf{k}) : g \in G\}$. G acts on $Gr(\mathbf{g}, \mathbf{k})$ via Ad action and this action is transitive. The isotropy group of the action at the point $\mathbf{k}$ is $N(K)$. We identify $Gr(\mathbf{g}, \mathbf{k})$ with $G/N(K)$. As we had mentioned earlier, since G/K is a rank-1 Riemannian symmetric space, $(G/K) \rightarrow [G/N(K)]$ is finite sheeted simply connected covering. Let us denote by $\tilde{Gr}(\mathbf{g}, \mathbf{k})$ the universal cover of $Gr(\mathbf{g}, \mathbf{k})$. $\tilde{Gr}(\mathbf{g}, \mathbf{k})$ can be identified with G/K . Because we are working with oriented foliations which we explained earlier, we have to take the space $\tilde{Gr}(\mathbf{g}, \mathbf{k})$ instead of $Gr(\mathbf{g}, \mathbf{k})$.

Let us denote by E the set of all possible tangent planes tangential to all admissible foliations $\mathcal{F}$ of G . Observe that E is a subbundle of the Grassmannian bundle of all n -dimensional subspaces of the tangent space at each point of G . We have the canonical projection from $\Pi_1 : E \rightarrow G$ and a natural map from $G \times Gr(\mathbf{g}, \mathbf{k}) \rightarrow E$ defined as

$$(g, Ad_h(\mathbf{k})) \mapsto L_{g*}(Ad_h(\mathbf{k})).$$

Let us denote by $\tilde{E}$ the universal cover of E . Then we see that this natural map lifts to a map from $G \times \tilde{Gr}(\mathbf{g}, \mathbf{k}) \rightarrow \tilde{E}$. Further $\tilde{E}$ is equivalent to $G \times \tilde{Gr}(\mathbf{g}, \mathbf{k})$ over G . If we recall that, for $x \in gKh^{-1}$ where $gKh^{-1} \in \mathcal{F}$, $T_x(gKh^{-1}) = L_{x*}(Ad_h(\mathbf{k}))$ the following map

$$\pi_2 : E \rightarrow \frac{G \times G}{I_K}$$

defined by

$$\pi_2(L_{g*}(Ad_h(\mathbf{k}))) = (\bar{g}\bar{h}, \bar{h})$$

is natural. This map π_2 can be lifted to map

$$\Pi_2 : E \rightarrow \frac{G}{K} \times \frac{G}{K}.$$

Again we remark that, we are lifting this map, because we are working with oriented foliations. Hence we get a map $\tilde{\Pi}_2 : \tilde{E} \rightarrow G/K \times G/K$ which is a K -bundle.

Following the notations of [1], we define, for $(\bar{g}_0, \bar{h}_0) \in (G/K) \times (G/K)$

$$\Delta(\bar{g}_0, \bar{h}_0) = \{(\bar{g}, \bar{h}) \in \frac{G}{K} \times \frac{G}{K} : d(\bar{g}, \bar{g}_0) = d(\bar{h}, \bar{h}_0)\}.$$

$\Delta(\bar{g}_0, \bar{h}_0)$ is called the *diagonal* centred at the point $(\bar{g}_0, \bar{h}_0)$. Then from lemma 2, we see that if $(\bar{g}, \bar{h}) \in \Delta(\bar{g}_0, \bar{h}_0)$, the submanifolds gKh^{-1} and $g_0Kh_0^{-1}$ intersect nontrivially. Hence if $(\bar{g}_0, \bar{h}_0) \in M_{\mathcal{F}}$ then $\Delta(\bar{g}_0, \bar{h}_0)$ intersects $M_{\mathcal{F}}$ only at $(\bar{g}_0, \bar{h}_0)$ and nowhere else.

We call the directions tangential to the fibres of the map Π_2 as horizontal directions and the ones tangential to the fibres of the map Π_1 as vertical. Since Π_2 is a projection in the second coordinate, $\Pi_1^{-1}(g) = g \times \tilde{G}(g, k)$ is mapped diffeomorphically on to its image, which is a diagonal centred at any point in the image of $\Pi_1^{-1}(g)$ under Π_2 , as can be seen easily.

Now we start proving the theorem.

Proof of Theorem 2. Let $\mathcal{F}$ be an oriented admissible foliation of G . From the construction of the bundle $\tilde{E}$, we have a section $S_{\mathcal{F}}$ of $\tilde{E}$ over G such that at each point $x \in G$, $S_{\mathcal{F}}(x)$ is the tangent plane tangential to the oriented admissible foliation $\mathcal{F}$ of G at that point, i.e. if $x \in gKh^{-1}$ for $gKh^{-1} \in \mathcal{F}$, then $s_{\mathcal{F}}(x) = T_x(gKh^{-1})$.

From the definition of $\tilde{\Pi}_2: \tilde{E} \rightarrow (G/K) \times (G/K)$, we see that

$$\tilde{\Pi}_2^{-1}(M_{\mathcal{F}}) = S_{\mathcal{F}}(G).$$

Let us now assume that f is differentiable and that $\|df\| < 1$. Then for any $(u, v) \in T_{(\tilde{g}, f(\tilde{g}))}(M_{\mathcal{F}})$, since $\|df\| < 1$, $\|u\| > \|v\|$. Hence (u, v) cannot be tangential to $\Delta(\tilde{g}, f(\tilde{g}))$. This means that $M_{\mathcal{F}}$ is nowhere tangential to diagonals is $(G/K) \times (G/K)$. Hence $\Pi_2^{-1}(M_{\mathcal{F}}) = S_{\mathcal{F}}(G)$ is a differentiable submanifold of $\tilde{E}$ which is nowhere tangential to vertical directions. This shows that $\Pi_1|_{S_{\mathcal{F}}(G)}: S_{\mathcal{F}}(G) \rightarrow G$ is diffeomorphism. Therefore the section $S_{\mathcal{F}}$ is differentiable and hence the foliation is differentiable.

Conversely, let us start with a differentiable foliation $\mathcal{F}$. This implies that $M_{\mathcal{F}} = \text{Graph}(f)$, for some strictly distance decreasing map $f: (G/K) \rightarrow (G/K)$, is a differentiable submanifold of $(G/K) \times (G/K)$. In general from the differentiability of the submanifold we cannot say the map is differentiable. Because the map is strictly distance decreasing, we can conclude that it is differentiable.

Now, we shall show that $\|df\| < 1$. f is strictly distance decreasing implies that $\|df\| \leq 1$. Since $S_{\mathcal{F}}$ is a differentiable section, $S_{\mathcal{F}}(G) = \tilde{\Pi}_2^{-1}(M_{\mathcal{F}})$ cannot be tangential to any vertical direction. But one can easily see that $S_{\mathcal{F}}(G) = \tilde{\Pi}_2^{-1}(M_{\mathcal{F}})$ is tangential to verticals iff $M_{\mathcal{F}}$ is tangential to diagonals $(G/K) \times (G/K)$. This coupled with the fact that $M_{\mathcal{F}}$ is tangential to some diagonal direction iff $\|df\| = 1$ at some point implies that $\|df\| < 1$. This finishes the proof. Q.E.D.

3. Topology of the space of foliations

In this section we shall describe the homotopy type of the space of foliations $\mathcal{F}$ of G , when $G/K = S^n$.

Let

$$\mathcal{D}\left(\frac{G}{K}\right) = \{f: G/K \rightarrow G/K: f \text{ is strictly distance decreasing}\}.$$

We observe that we have a one-one map from $\mathcal{D}(G/K)$ on to the space of all admissible foliations $\mathcal{F}$ of G with fixed orientation. Hence we study the space $\mathcal{D}(G/K)$ with compact open topology to understand the space of all admissible oriented foliations

Theorem 3. $\mathcal{D}(G/K)$ is path connected and deformation retracts to the space of constant maps when $G/K = S^n$.

Remark. The proof uses Borsuk-Ulam theorem as in [1]. We do not have an analogous result for other rank-1 Riemannian symmetric spaces. Even if we do not have Borsuk-Ulam type of result, it is enough if we can prove that the image of G/K for any $f \in \mathcal{D}(G/K)$, is contained in a ball of radius less than half the diameter of G/K . For, given any $f \in \mathcal{D}(G/K)$ the cut-locus of any point $f(x)$ is not in the image of f . This tells us that the image has diameter less than $\pi/2$ where $\text{diam}(G/K) = \pi/2$. Since two geodesics starting from a point diverge up to a distance $\pi/4$, we can use the radial contraction along geodesics starting from the centre of the ball, provided we prove that the image lies inside a ball of radius less than $\pi/4$.

Proof. First we shall prove that $\mathcal{D}(G/K)$ is path connected when $G/K = S^n$. Let $f \in \mathcal{D}(G/K)$. Since f is strictly distance decreasing, $f(G/K)$ cannot be the whole of G/K . Then by Borsuk-Ulam theorem there exists a pair of antipodal points (x_+, x_-) in S^n such that $f(x_+) = f(x_-) = y$. This shows that $\text{Im}(f)$ lies inside a ball of radius less than $\pi/2$ where the metric on S^n is normalized such that $\text{diam } S^n = \pi$. Then we use radial contraction along geodesics issuing from y , to homotope f through distance decreasing maps to the constant map y , the centre of the hemisphere containing the image of f . This proves that $\mathcal{D}(G/K)$ is path connected when $G/K = S^n$.

Claim. There is a continuous map $c: \mathcal{D}(G/K) \rightarrow G/K$ such that $\text{Im}(f)$ for each $f \in \mathcal{D}(G/K)$ is contained in the open hemisphere centred at $c(f)$.

The proof follows along the same lines as that of S^2 for which a proof is given in Lemma 9.3 of [1]. Hence we omit the proof here. Q.E.D.

Once we have the theorem the following corollary is immediate.

COROLLARY 1.

The space of all admissible foliations has exactly $\#I_K/(K \times K)$ path components.

Thus the space of all admissible oriented foliations has the homotopy type of disjoint union of $\#I_K/(K \times K)$ copies of G/K .

Theorem 4. *Let $(G/K) = S^n$. Then every oriented admissible K -foliation contains two parallel leaves.*

Proof. In Theorem C of [1] it is proved that any great circle fibration of S^3 must contain some orthogonal pair of circles. We remark that orthogonal circles are parallel and conversely. Hence we refer to Theorem C of [1], as the proof follows along the same lines as that of S^2 . Q.E.D.

Concluding remarks

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On infinite-dimensional control systems with state and control constraints

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Abstract. In this paper we examine infinite-dimensional control systems governed by semilinear evolution equations and having both state and control constraints. We introduce the relaxed system and show that the original trajectories are dense in an appropriate function space in the relaxed ones. We also determine the dependence of the solution set on the initial conditions. Then using those results we establish necessary and sufficient conditions for optimality for some optimization problems. Finally we prove some controllability results.

Keywords. Relaxation; relaxed system; lower semicontinuous and upper semicontinuous multifunction; continuous selections; convex subdifferential; controllability.

1. Introduction

In this paper we consider control systems governed by semilinear evolution equations, with both state and control constraints. The study of infinite-dimensional control systems was initiated by the works of Egorov [8] and Lions [13] and since then it has attracted the interest of many mathematicians and has become one of the most important and productive areas of applied mathematics. The majority of the works treats systems with no state constraints and with time independent control constraints. Recently Barbu [2] and Mackenroth [15], studied convex optimal control problems where a linear equation and a state constraint are given.

In this work, we study systems governed by a semilinear evolution equation and having general state and control constraints of the form $x(t) \in K(t)$ and $u(t) \in U(t)$, where $K(\cdot)$, $U(\cdot)$ are the time varying sets (they can be functional constraints or inclusion constraints or combinations thereof). We define the relaxed (convexified) system, and we address the “relaxation” problem, namely we examine what is the relation between the sets of trajectories of the original and relaxed systems. Then we examine the dependence of the set of solutions of the original system on the initial condition. Combining these results, we are then able to obtain some necessary and sufficient conditions for the solutions of some nonlinear, convex optimization problems and determine some continuity properties of the corresponding value (Bellman) function. Finally we derive some necessary and sufficient conditions for controllability.

2. Preliminaries

Let X be a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(w)}(X) = \{A \subseteq X: \text{nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)} = \{A \subseteq X: \text{nonempty, (w-) compact, (convex)}\}.$$

If (Ω, Σ) is a measurable space and $F: \Omega \rightarrow 2^{P_{f(w)}(X)}$, we say that F is graph measurable, if $\text{Gr } F = \{(w, x) \in \Omega \times X: x \in F(w)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel σ -field of X .

If Y, Z are Hausdorff topological spaces, a multifunction $G: Y \rightarrow 2^{Z}$ is said to be upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if for every $C \subset Z$ closed $G^+(C) = \{y \in Y: G(y) \cap C \neq \emptyset\}$ (resp. $G^-(C) = \{y \in Y: G(y) \subset C\}$) is closed in Y . Under additional hypotheses on Y, Z we can have other equivalent definitions of upper and lower semicontinuity. For details we refer to Delahaye-Denel [4]. When a multifunction is both u.s.c. and l.s.c., then it is said to be continuous.

On $P_f(X)$, we can define a generalized metric, known as the "Hausdorff metric" by setting:

$$h(A, B) = \max \{\sup \{d(a, B), a \in A\}, \sup \{d(b, A), b \in B\}\}.$$

Here $A, B \in P_f(X)$ and $d(a, B) = \inf \{\|a - b\|: b \in B\}$, the distance of a from B (similarly for $d(b, A)$). It is well known that $(P_f(X), h)$ is a complete metric space. A multifunction $F: X \rightarrow P_f(X)$ is said to be Hausdorff continuous (h -continuous), if it is continuous from X into the metric space $(P_f(X), h)$. In general, continuity and h -continuity are not comparable notions. However they coincide if the multifunction is compact valued (i.e. has values in $P_k(X)$). This is a consequence of the fact that on $P_k(X)$ the Vietoris and Hausdorff topologies coincide (see Klein-Thompson [10], p. 41).

Now let $f: X \rightarrow \mathbb{R} = \mathbb{R}U\{+\infty\}$ be a proper (i.e. not identically $+\infty$), convex function. Let $x \in \text{dom } f = \{x \in X: f(x) < +\infty\}$. The subdifferential of f at x is defined to be the set

$$\partial f(x) = \{x^* \in X^*: (x^*, x' - x) \leq f(x') - f(x), x' \in X\}$$

(see Laurent [12]).

Finally if $A \subseteq X$, then the support function of A is the function $\sigma(\cdot, A): X^* \rightarrow \mathbb{R}$ defined by

$$\sigma(x^*, A) = \sup \{(x^*, x): x \in A\},$$

where $(\cdot, \cdot)$ denotes the duality brackets for the pair (X, X^*) .

3. Relaxation results

space), we have $X \hookrightarrow H \hookrightarrow X^*$ with all embeddings continuous and dense. We will also assume that they are compact. To have an example in mind consider $X = H_0^1(0, 1)$, $H = L^2(0, 1)$ and $X^* = H^{-1}(0, 1)$. Such a triple of spaces (X, H, X^*) is often called "Gelfand triple" or "spaces in normal position". By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of X, X^*). Also by $(\cdot, \cdot)$ we will denote the inner product in H and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X, X^*) . The two are compatible in the sense that if $x \in X \subseteq H$ and $h \in H \subseteq X^*$, then we have $(x, h) = \langle x, h \rangle$. By H_w we will denote the space H endowed with the weak topology. Also let Z be a separable Banach space, modelling the control space. The distributed parameter control system under consideration is the following:

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t)x(t) = f(t, x(t), u(t)) \text{ a.e. on } T \\ x(0) = x_0, x(t) \in K(t), t \in T \\ u(t) \in U(t) \text{ a.e., } u(\cdot) \text{ is measurable} \end{array} \right\} (*).$$

To this system we associate a larger system, with convexified dynamics, known as the "relaxed system".

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t)x(t) = \int_Z f(t, x(t), z) \lambda(t)(dz) \\ x(0) = x_0, x(t) \in K(t), t \in T \\ \lambda(t) \in \Sigma(t) \text{ a.e., } \lambda(\cdot) \text{ is measurable} \end{array} \right\} (*).$$

Here $\Sigma(t) = \{\lambda \in M_+^1(Z) : \lambda(U(t)) = 1\}$ (by $M_+^1(Z)$ we denote the space of probability measures on Z). So $\lambda(\cdot)$ is a transition probability. Note that in the dynamical equation governing $(*)$, the control function $\lambda(\cdot)$ enters linearly.

We will make the following hypotheses concerning the data of our problems.

(A): $A: T \times X \rightarrow X^*$ is an operator s.t.

- (1) $A(t)(\cdot)$ is linear, continuous,
- (2) $\|A(t')x - A(t)x\|_* \leq k|t' - t|\|x\|, k > 0$ for all $t', t \in T, x \in X$,
- (3) $\langle A(t)x, x \rangle \geq c\|x\|^2, c > 0$ (i.e. $A(t)(\cdot)$ is strongly monotone).

(f): $f: T \times H \times Z \rightarrow H$ is a function s.t.

- (1) $t \rightarrow f(t, x, z)$ is measurable,
- (2) $x \rightarrow -f(t, x, z)$ is continuous, monotone,
- (3) $(x, z) \rightarrow f(t, x, z)$ is continuous from $H \times Z$ into H_w ,
- (4) $|f(t, x, z)| \leq a(t) + b(t)(|x| + \|z\|)$ a.e., with $a(\cdot), b(\cdot) \in L_+^2$.

Remark. The dissipativity hypothesis $H(f)$ (2) on $f(t, \cdot, z)$ can be replaced by

- (2)' $|f(t, x', z) - f(t, x, z)| \leq k(t)|x' - x|$ a.e., with $k(\cdot) \in L_+^1$.

(U): $U: T \rightarrow P_{fc}(Z)$ is a graph measurable multifunction s.t. $U(t) \subseteq W$ a.e., where $W \in P_{wkc}(Z)$.

By Proposition 5.5.1 of Tanabe [22], we know that

the dynamics of (*), is given by the following variation of constants formula:

$$x(t) = S(t, 0)x_0 + \int_0^t S(t, s)f(s, x(s), u(s)) ds, \quad t \in T.$$

It is well known (see for example Barbu [3], p. 167 or Lions [14]), that in this case $x(\cdot) \in W(T) = \{x \in L^2(X); \dot{x} \in L^2(X^*)\} \subseteq C(T, H)$.

We will make the following hypothesis on $S(t, s)$: H_c : $S(t, s)$ is compact for $t > s$, $(t, s) \in \Delta$.

To have a typical example in mind consider the Laplacian.

Let $Q(x_0)$ be the set of trajectories of (*) and $Q_r(x_0)$ the set of trajectories of (*),. Note that given any initial control function $u(\cdot)$, we can view it as a relaxed control by considering $\delta(\cdot)$, the Dirac transition probability. So $Q(x_0) \subseteq Q_r(x_0)$. In [20], with no state constraints present (i.e. $K(t) = H$ for all $t \in T$), we proved that under hypotheses $H(A)$, $H(f)$, $H(U)$, H_c , we have $\overline{Q(x_0)} = \overline{Q_r(x_0)}$, the closure taken in $C(T, H)$. We would like to have the same result, in the presence of the state constraints $K(\cdot)$. For this we will need the following hypothesis on $K(\cdot)$.

$H(K)$: $K: T \rightarrow P_{rc}(H)$ is l.s.c. and $\text{int } K(t) \neq \emptyset$ for all $t \in T$.

A well known selection theorem of Michael [16], tells us that under the above hypothesis, there exists $k: T \rightarrow H$ continuous s.t. $k(t) \in K(t)$ for all $t \in T$ (i.e. $K(\cdot)$ admits a continuous selector). Denote by $CS(K)$ the set of continuous selectors of $K(\cdot)$.

To prove our density (relaxation) result, we will need the following three auxiliary results, which are actually of independent interest.

Lemma 1. *If hypothesis $H(K)$ holds and $x(\cdot) \in CS(\text{int } K)$ then $t \rightarrow p(t) = d(x(t), \text{bd } K(t))$ is l.s.c. where $\text{bd } K(t)$ denotes the boundary of $K(t)$.*

Remark. Because of Michael's theorem [16] (theorem 3.2"), we know that $CS(\text{int } K) \neq \emptyset$. Also note that even though $K(\cdot)$ is lower semicontinuous, $\text{bd } K(\cdot)$ need not be such. For example let $K(t) = [0, 1]$ if $t \neq 0$ and $K(0) = [0, 1/2]$. Clearly $K(\cdot)$ is l.s.c. but $\text{bd } K(\cdot)$ is neither l.s.c. nor u.s.c.

Proof. We need to show that for all $\lambda > 0$, the level set

$$L(\lambda) = \{t \in T: p(t) \leq \lambda\}$$

is closed in T . So let $t_n \rightarrow t$, $t_n \in L(\lambda)$, $n \geq 1$. Let $\varepsilon > 0$, $\varepsilon < \lambda$ be given. Due to the lower semicontinuity of $K(\cdot)$ (hypothesis $H(K)$), we can find $n_0 \geq 1$ s.t., for $n \geq n_0$ we have:

$$K(t) \subseteq K(t_n) + \hat{B}_\varepsilon,$$

where $\hat{B}_\varepsilon$ is the open ε -ball in H . Also there exists $n_1 \geq 1$ s.t. for $n \geq n_1$ we have $|x(t_n) - x(t)| < \varepsilon$. Then for $z \in \text{bd } K(t)$ and for $n \geq n_2 = \max(n_0, n_1)$, we have: $p(t) \leq |x(t) - z| \leq |x(t) - x(t_n)| + |x(t_n) - z|$. Therefore $p(t) < \varepsilon + d(x(t_n), \text{bd } K(t))$. Since

From this inequality we see that if there exists $\{t_{n_k}\}_{k \geq 1} \subseteq \{t_n\}_{n \geq 1}$ s.t.

$$d(x(t_{n_k}), bd(K(t_{n_k}) + \hat{B}_\varepsilon)) \leq \lambda,$$

then in the limit as $k \rightarrow \infty$, we will get

$$p(t) < \varepsilon + \lambda.$$

Let $\varepsilon \downarrow 0$. We have $p(t) \leq \lambda \Rightarrow t \in L(\lambda)$. So assume that for all but finitely many $n \geq n_2$ we have:

$$d(x(t_n), bd(K(t_n) + \varepsilon \hat{B}_1)) = \hat{\lambda}_n > \lambda.$$

This inequality means that $x(t_n) + \hat{B}_{\hat{\lambda}_n - \varepsilon} \subseteq K(t_n) + \hat{B}_\varepsilon \subseteq K(t_n) + \hat{B}_\varepsilon$ for $n \geq n_2$. So for $n \geq n_2$ we have

$$x(t_n) + \hat{B}_{\hat{\lambda}_n - \varepsilon} \subseteq K(t_n) \Rightarrow d(x(t_n), bdK(t_n)) \geq \hat{\lambda}_n - \varepsilon$$

$$\hat{\lambda}_n \leq \varepsilon + d(x(t_n), bdK(t_n)) = \varepsilon + p(t_n) \Rightarrow p(t) < 2\varepsilon + p(t_n) < 2\varepsilon + \lambda.$$

Let $\varepsilon \downarrow 0$. We get $p(t) \leq \lambda \Rightarrow t \in L(\lambda)$ is closed $\Rightarrow p(\cdot)$ is l.s.c..

Q.E.D.

Remark. It is clear from the above proof that H can be any Banach space (note that since T is a compact metric space we do not need separability on H ; see Michael [16], theorem 3.2").

Lemma β . If hypothesis $H(K)$ holds, then $CS(\text{int } K) = \text{int } CS(K) \neq \emptyset$.

Proof. It is clear from the definitions that $\text{int } CS(K) \subseteq CS(\text{int } K)$. We will now show that the opposite inclusion also holds. Let $x(\cdot) \in CS(\text{int } K)$ and set $p(t) = d(x(t), bdK(t))$. From lemma α , we know that $p(\cdot)$ is l.s.c.. Since T is compact, there exists $t \in T$ s.t.

$$\inf_{t \in T} p(t) = p(\hat{t}) = d(x(\hat{t}), bdK(\hat{t})) > 0$$

since $x(\hat{t}) \in \text{int } K(\hat{t})$. Let $\delta < p(\hat{t})$. Then clearly $x(\cdot) + \hat{B}_\delta \subseteq CS(K) \Rightarrow x(\cdot) \in \text{int } CS(K) \Rightarrow CS(\text{int } K) = \text{int } CS(K)$ and the nonemptiness follows from proposition 2.3 and theorem 3.1''' (c) of Michael [16].

Q.E.D.

Remark. Again H can be any Banach space.

For the next lemma assume that Y is any Banach space.

Lemma γ . If $A, B \subseteq Y$ are such that $\bar{A}$ is convex, B is convex with $\text{int } B \neq \emptyset$,

$$\text{and } \bar{A} \cap \text{int } B \neq \emptyset, \\ \text{then } \bar{A} \cap B = \bar{A} \cap \bar{B}.$$

Proof. From lemma 5 of Moreau [17], we know that $\sigma_{\bar{A} \cap \text{int } B}(\cdot) = \sigma_{\bar{A} \cap \bar{B}}(\cdot) = (\sigma_A \square \sigma_B)(\cdot)$ (the infimal convolution in the sense of convex analysis of $\sigma_A(\cdot)$ and $\sigma_B(\cdot)$, see Laurent [12]). Since $\bar{A} \cap \text{int } B$ and $\bar{A} \cap \bar{B}$ are both convex, from the equality of their support

functions, we get that $\overline{A \cap \text{int } B} = \overline{A} \cap \overline{B}$. Let $x \in \overline{A} \cap \text{int } B$. Then we can find $x_n \in A$ s.t. $x_n \rightarrow x$ and $\delta > 0$ s.t. $x + \hat{B}_\delta \subseteq \text{int } B$. So for $n \geq 1$ large enough, we will have that $x_n \in x + \hat{B}_\delta \Rightarrow x_n \in A \cap \text{int } B \Rightarrow x \in \overline{A \cap \text{int } B} \Rightarrow \overline{A} \cap \text{int } B = \overline{A} \cap \text{int } B = \overline{A} \cap \overline{B} \Rightarrow \overline{A \cap B} = \overline{A} \cap \overline{B}$ as claimed.

Q.E.D.

Now we are ready for the theorem comparing the sets $Q(x_0)$ and $Q_r(x_0)$.

Theorem 3.1. *If hypotheses $H(A)$, $H(f)$, $H(U)$, H_c and $H(K)$ hold and there exists $x(\cdot) \in Q(x_0)$ s.t. $x(t) \in \text{int } K(t)$ for all $t \in T$, then $Q_r(x_0) = \overline{Q(x_0)}$ the closure taken in $C(T, H)$.*

Proof. Let $P(x_0)$ and $P_r(x_0)$ be the sets of trajectories of $(*)$ and $(*)_r$, respectively, when no state constraints are present (i.e. $K(t) = H$ for all $t \in T$). From theorem 3.2 of [20], we know that $P_r(x_0) = \overline{P(x_0)}$, the closure in $C(T, H)$. Now observe that:

$$Q(x_0) = P(x_0) \cap CS(K)$$

and

$$Q_r(x_0) = P_r(x_0) \cap CS(K).$$

So $Q_r(x_0)$ is convex, closed in $C(T, H)$. From our interiority hypothesis and lemma γ , we have that $\text{int } CS(K) \neq \emptyset$. Since $\overline{P(x_0)} = P_r(x_0)$ is convex, $CS(K)$ is convex with nonempty interior and by hypothesis $P(x_0) \cap \text{int } CS(K) \neq \emptyset$, we can apply lemma β and get that $\overline{Q(x_0)} = \overline{P(x_0) \cap CS(K)} = \overline{P(x_0)} \cap \overline{CS(K)} = P_r(x_0) \cap CS(K) = Q_r(x_0)$, the closure in $C(T, H)$.

Q.E.D.

An immediate, useful consequence of the above density result is the following theorem concerning the reachable sets $R(t) = \{x(t); x(\cdot) \in Q(x_0)\}$ and $R_r(t) = \{x(t); x(\cdot) \in Q_r(x_0)\}$, $t \in T$.

Theorem 3.2. *If the hypotheses of theorem 3.1 hold, then for all $t \in T$, $\overline{R(t)} = R_r(t)$, the closure taken in the strong topology of H .*

Proof. Note that for every $t \in T$, $R(t) = e_t(Q(x_0))$ and $R_r(t) = e_t(Q_r(x_0))$, where $e_t(\cdot)$ is the evaluation of $t \in T$ map. Recalling that $e_t(\cdot)$ is continuous on $C(T, H)$ (see for example Kuratowski [11]) and using theorem 3.1, we get:

$$e_t(Q_r(x_0)) = e_t(\overline{Q(x_0)}) \subseteq \overline{e_t(Q(x_0))}.$$

But from [20] we know that $Q_r(x_0)$ is compact in $C(T, H)$. So we have:

$$e_t(Q_r(x_0)) = \overline{e_t(Q(x_0))}$$

$$\Rightarrow R_r(t) = \overline{R(t)}, \text{ the closure taken in the strong topology on } H.$$

Q.E.D.

Next we will examine the dependence of the set $Q_r(x_0)$ on the initial condition x_0 . Such a study will lead us to some interesting observations concerning the value

Proof. We need to show that for all $C \subseteq Y$ nonempty, closed, we have that $G^-(C) = \{x \in V: G(x) \cap C \neq \emptyset\}$ is closed too. So let $\{x_n\}_{n \geq 1} \subseteq G^-(C)$, $x_n \rightarrow x$ in V . Then by hypothesis $G(\cdot)$ restricted on $K = \{x_n, x\}_{n \geq 1}$ is u.s.c.. Thus $G(K)$ is compact (see Klein–Thompson [10], theorem 7.4.2, p. 90). Let $y_n \in G(x_n) \cap C$. Since $\{y_n\}_{n \geq 1} \subseteq G(K)$, by passing to a subsequence if necessary, we may assume that $y_n \rightarrow y$ in Y . So $y \in \lim G(x_n)$ and because $G|_K$ is u.s.c., we have $\lim G(x_n) \subseteq G(x) \Rightarrow y \in G(x)$ and $y \in C \Rightarrow G(x) \cap C \neq \emptyset \Rightarrow x \in G^-(C) \Rightarrow G(\cdot)$ is indeed u.s.c.

Q.E.D.

Having this lemma, we can prove our first theorem concerning the dependence of $Q_r(x_0)$ on the initial condition x_0 .

In what follows we set $F(t, x) = \cup \{f(t, x, u): u \in U(t)\}$ and $F_r(t, x) = \{\int_W f(t, x, z) \lambda(dz): \lambda \in \Sigma(t)\}$. From the proof of theorem 3.2 in [20] we know that $\overline{\text{conv}} F(t, x) = F_r(t, x)$. So $F_r(t, x) \in P_{\text{wkc}}(H)$ for all $(t, x) \in T \times H$.

Theorem 3.3. *If the hypotheses of theorem 3.1 hold for all $x_0 \in V \subseteq H$, then $Q_r: V \rightarrow P_k(C(T, H))$ is u.s.c.*

Proof. In view of lemma δ and the equivalences in Delahaye–Denel [4], we only need to show that if $x_0^n \xrightarrow{s} x_0$ in V , then $\overline{\lim} Q_r(x_0^n) \subseteq Q_r(x_0)$.

So let $y \in \overline{\lim} Q_r(x_0^n)$. Then by definition (see Kuratowski [11], p. 337), we can find $y_{n_k} \in Q_r(x_0^{n_k})$ s.t. $y_{n_k} \xrightarrow{s} y$ in $C(T, H)$. Then we have:

$$y_{n_k}(t) = S(t, 0)x_0^{n_k} + \int_0^t S(t, s)g_{n_k}(s) ds, \quad t \in T$$

where

$$g_{n_k}(\cdot) \in S_{F_r}^1(\cdot, y_{n_k}(\cdot)) = \{h \in L^1(H): h(t) \in F_r(t, y_{n_k}(t)) \text{ a.e.}\}.$$

From proposition 1, p. 47, of Aubin–Cellina [1], we know that $F(t, \cdot)$ is u.s.c. from H into H_w . So $F_r(t, \cdot)$ is u.s.c. from H into H_w . Therefore we have that $\overline{\text{conv}} \bigcup_{k \geq 1} F(t, y_{n_k}(t)) = L(t) \in P_{\text{wkc}}(H)$. Furthermore for every $k \geq 1$ we have:

$$\begin{aligned} \text{Gr } F(\cdot, y_{n_k}(\cdot)) &= \{(t, x) \in T \times H: x \in F(t, y_{n_k}(t))\} \\ &= \{(t, x) \in T \times H: x = f(t, y_{n_k}(t), u), u \in U(t)\} \\ &= \{(t, x) \in T \times H: x - f(t, y_{n_k}(t), u) = 0, d(u, U(t)) = 0\}. \end{aligned}$$

Let $\theta(t, x, u) = x - f(t, y_{n_k}(t), u)$ and $\eta(t, u) = d(u, U(t))$. From $H(f)$ we have that $\theta(\cdot, \cdot, \cdot)$ is measurable on $T \times H \times W_w$, where W_w is the set W with the relative weak topology. Also from $H(U)$ we have that $\eta(\cdot, \cdot)$ is measurable on $T \times W$. Furthermore from corollary 2.4 of Edgar [7], we have that $B(Y_w) = B(Y) \Rightarrow B(Y_w) \cap W = B(Y) \cap W \Rightarrow B(W_w) = B(W)$. Hence

$$\{(t, x, u) \in T \times H \times W: \theta(t, x, u) = 0, \eta(t, u) = 0\} \in B(T) \times B(X) \times B(W_w).$$

But recall that W_w is a compact Polish space (see Dunford–Schwartz [6], theorem

3, p. 434). So we can apply the Arsenin–Novikov theorem (see Dellacherie [5]) and get that:

$$\begin{aligned} \text{proj}_{T \times X} \{ (t, x, u) \in T \times H \times W : \theta(t, x, u) = 0, \eta(t, u) = 0 \} &\in B(T) \times B(H) \\ \Rightarrow \text{Gr } F(\cdot, y_{n_k}(\cdot)) &\in B(T) \times B(H) \Rightarrow F_r(\cdot, y_{n_k}(\cdot)) \end{aligned}$$

is graph measurable $\Rightarrow L(\cdot)$ is graph measurable. Also from $H(f)$, it is easy to see that $t \rightarrow |L(t)| = \sup \{ |y| : y \in L(t) \} \in L^1_+$. So we can apply proposition 3.1 of [18] and deduce that by passing to a subsequence if necessary, we may assume that

$g_{n_k} \xrightarrow{w} g$ in $L^1(H)$. Then from theorem 3.1 of [19], we have:

$$\begin{aligned} g(t) &\in \overline{\text{conv } w\text{-}\lim F_r(t, y_{n_k}(t))} \subseteq F_r(t, y(t)) \text{ a.e.} \\ &\Rightarrow g \in S^1_{F_r}(\cdot, y(\cdot)) \\ &\Rightarrow y(t) = S(t, 0)x_0 + \int_0^t S(t, s)g(s) \, ds, \quad t \in T. \end{aligned}$$

Let $L(t) = \{ \lambda \in \Sigma(t) : g(t) = \int_W f(t, y(t), z)\lambda(dz) \}$. From the definition of $F_r(\cdot, \cdot)$ it is clear that $L(t) \neq \emptyset$ for all $t \in T$. Let $p : T \times M^1_+(W_w) \rightarrow H$ be defined by $p(t, \lambda) = g(t) - \int_W f(t, y(t), z)\lambda(dz)$. Clearly $p(t, \lambda)$ is measurable in t , continuous in λ . Hence since $M^1_+(W_w)$ with the usual narrow (weak) topology, is compact metrizable, we conclude that $p(t, \lambda)$ is jointly measurable. Therefore since $\text{Gr } \Sigma \in B(T) \times B(M^1_+(W_w))$ (see [20]), we conclude that:

$$\text{Gr } L = \{ (t, \lambda) \in T \times M^1_+(W_w) : p(t, \lambda) = 0 \} \cap \text{Gr } \Sigma \in B(T) \times B(M^1_+(W_w)).$$

Apply Aumann's selection theorem (see Wagner [23]) to get $\lambda : T \rightarrow M^1_+(W_w)$ measurable s.t. $\lambda(t) \in L(t)$ for all $t \in T$. Then $g(t) = \int_W f(t, y(t), z)\lambda(t)(dz) \Rightarrow y \in Q_r(x_0) \Rightarrow Q_r(\cdot)$ is indeed u.s.c. from H into $P_k(C(T, H))$.

Q.E.D.

For a simpler form of the system, in which state and control functions are separated, we can say more.

So consider the following "original" and "relaxed" systems:

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t)x(t) = f(t, u(t)) \text{ a.e. on } T \\ x(0) = x_0, x(t) \in K(t) t \in T, \\ u(t) \in U(t) \text{ a.e., } u(\cdot) \text{ is measurable} \end{array} \right\}^{(**)}$$

and

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t)x(t) = \int_W f(t, z)\lambda(t)(dz) \text{ a.e.} \\ x(0) = x_0, x(t) \in K(t) t \in T, \\ \lambda(t) \in \Sigma(t) \text{ a.e., } \lambda(\cdot) \text{ is measurable} \end{array} \right\}^{(**)}.$$

As before $\Sigma(t) = \{ \lambda \in M^1_+(W_w) : \lambda(U(t)) = 1 \}$ and $Q(x_0)$ (resp. $Q_r(x_0)$) is the solution set of $(**)$ (resp. $(**)_r$). Hypothesis $H(f)$ now takes the following simpler form: $H(f)_1 : f : T \times Z \rightarrow H_w$ is a map s.t.

- (1) $t \rightarrow f(t, z)$ is continuous,
- (2) $z \rightarrow f(t, z)$ is continuous from Z into H_w ,
- (3) $|f(t, z)| \leq \alpha(t)$ a.e. with $\alpha(\cdot) \in L^2$

Theorem 3.4. *If hypotheses $H(A)$, $H(f)_1$, $H(U)$, H_c , $H(K)$ hold and there exists*

$$\hat{x}(\cdot) \in Q(x_0) \text{ s.t. } x(t) \in \text{int } K(t) \text{ for all } t \in T,$$

then there exists $\delta > 0$ s.t. for all $x'_0 \in (x_0 + \hat{B}_\delta) = V$, $Q(x'_0) \neq \emptyset$ and $Q(\cdot)$ is l.s.c. from V into $2^{C(T,H) \setminus \{\emptyset\}}$.

Proof. Let $\hat{x}(\cdot)$ be the interior trajectory emanating from x_0 , postulated by our hypothesis. Let $\hat{u}(\cdot) \in S_V^1$ be the admissible control generating $\hat{x}(\cdot)$. From lemma β we know that there exists $\delta_1 > 0$ s.t. $\hat{x} + \hat{B}_{\delta_1} \subseteq CS(K)$. Let $\delta = (\delta_1/M)$ (where $\|S(t, s)\| \leq M$ for all $(t, s) \in \Delta$) and for $x'_0 \in x_0 + \hat{B}_{\delta_1}$ define:

$$\begin{aligned} x(t) &= S(t, 0)x'_0 + \int_0^t S(t, s)f(s, \hat{u}(s)) \, ds \\ \Rightarrow \|x(t) - \hat{x}(t)\| &\leq \|S(t, 0)\| \cdot \|x'_0 - x_0\| < \delta \\ \Rightarrow x(\cdot) &\in P(x'_0) \cap CS(K) = Q(x'_0) \\ \Rightarrow Q(x'_0) &\neq \emptyset \text{ for all } x'_0 \in x_0 + \hat{B}_\delta. \end{aligned}$$

Next let $x(\cdot) \in Q(x_0)$. From lemma γ we know that we can find $z_\varepsilon(\cdot) \in P(x_0) \cap \text{int } CS(K)$ (i.e. an interior trajectory) s.t. $\|x - z_\varepsilon\|_\infty < \varepsilon/2$ in $C(T, H)$. Also from above we know that we can find $\delta > 0$ s.t. for all $x'_0 \in x_0 + \hat{B}_\delta$ there exists $y(\cdot) \in Q(x'_0)$ s.t. $\|z_\varepsilon - y\|_\infty < \varepsilon/2$ in $C(T, H)$. Therefore $\|x - y\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow x(\cdot) \in Q(x'_0) + \hat{B}_\varepsilon$. Since $x(\cdot) \in Q(x_0)$ was arbitrary, we get $Q(x_0) \subseteq Q(x'_0) + \hat{B}_\varepsilon$ for all $x'_0 \in x_0 + \hat{B}_\delta \Rightarrow Q(\cdot)$ is l.s.c. on V .

Q.E.D.

Combining theorems 3.3 and 3.4 we get:

Theorem 3.5. *If the hypotheses of theorem 3.4 hold for all $x_0 \in V \subseteq H$, then $Q_r: V \rightarrow P_k(C(T, H))$ is h -continuous.*

Proof. From theorems 3.3 and 3.4 we get that $Q_r(\cdot)$ is continuous on V and because it is $P_k(C(T, H))$ -valued, it is h -continuous, as claimed.

Q.E.D.

4. Optimization problems

In this section we solve some optimization problems involving the systems considered in §3.

So let $\phi: H \times C(T, H) \rightarrow \mathbb{R}$ be a continuous function s.t. for every $x_0 \in H$, $\phi(x_0, \cdot)$ is convex. We consider the following optimization problem:

$$(O_1): v(x_0) = \inf \{ \phi(x_0, x): x(\cdot) \in Q(x_0) \}.$$

Proof. Let $\delta_{Q(x_0)} : C(T, H) \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by $\delta_{Q(x_0)}(x) = 0$ if $x \in Q(x_0)$, and $+\infty$ otherwise. Exploiting the continuity of $\phi(\cdot, \cdot)$ we have:

$$\inf_{x \in Q(x_0)} \phi(x_0, x) = \inf_{x \in \overline{Q(x_0)}} \phi(x_0, x) = \inf_{x \in C(T, H)} [\phi(x_0, x) + \delta_{\overline{Q(x_0)}}(x)].$$

From theorem 3.1 we know that $\overline{Q(x_0)} = Q_r(x_0)$ and the latter is convex. So $\delta_{\overline{Q(x_0)}}(\cdot)$ is a closed convex function. Hence our problem is a convex optimization problem. So the optimality condition from convex analysis (see Laurent [12], theorem 6.4.5, p. 352), tells us that $\hat{x} \in Q(x_0)$ is a solution if and only if $0 \in \partial[\phi(x_0, \hat{x}) + \delta_{\overline{Q(x_0)}}(\hat{x})]$. Since $\phi(x_0, \cdot)$ is continuous we can apply the Moreau–Rockafellar theorem (see Laurent [12], theorem 6.6.7, p. 369) and get that $\partial[\phi(x_0, \hat{x}) + \delta_{\overline{Q(x_0)}}(\hat{x})] = \partial\phi(x_0, \hat{x}) + \partial\delta_{\overline{Q(x_0)}}(\hat{x})$. Therefore there exists $x^* \in \partial_2\phi(x_0, \hat{x})$ s.t. $-x^* \in \partial\delta_{\overline{Q(x_0)}}(\hat{x}) \Rightarrow ((-x^*, \hat{x})) = \sigma(-x^*, \overline{Q(x_0)}) \Rightarrow ((x^*, \hat{x})) = \inf\{((x^*, x)); x \in Q(x_0)\}$.

The last part of the theorem, concerning the upper semicontinuity of the value functional $v(\cdot)$, follows from theorem 3.3 of this paper and theorem 5, p. 52, of Aubin–Cellina [1].

Q.E.D.

Remark. If $\phi(x_0, x) = \phi(x) = \int_0^b f(t, x(t)) dt$, then the conclusion of the theorem takes the following form: " $\hat{x}(\cdot) \in Q(x_0)$ solves (O_1) if and only if there exists $x^*(\cdot) \in M(T, H)$ s.t. $x^*(t) \in \partial f(t, \hat{x}(t))$ a.e. and $((x^*, \hat{x})) = \min\{((x^*, x)); x \in Q(x_0)\}$. This is a consequence of a result of Rockafellar [21] (theorem 22), which says that $x^* \in \partial\phi(\hat{x}) \Leftrightarrow x^*(t) \in \partial f(t, \hat{x}(t))$ a.e.

If instead we consider system (**), then we can get a stronger result concerning the value function $v(\cdot)$.

Theorem 4.2. *If the hypotheses of theorem 3.5 hold, then $v : V \rightarrow \mathbb{R}$ is continuous.*

Proof. This follows from theorem 3.5, combined with theorem 6, p. 53 of Aubin–Cellina [1].

Q.E.D.

Now we pass to a terminal cost control problem. So consider the following optimization problem:

$$(O_2) \inf\{\psi(x(b)); x(\cdot) \in Q(x_0)\}.$$

Here $\psi : H \rightarrow \mathbb{R}$ is continuous, convex. We have the following “minimum principle”, characterizing the solutions of (O_2) .

Theorem 4.3. *If the hypotheses of theorem 3.1 hold, then $\hat{x}(\cdot) \in Q(x_0)$ solves (O_2) if and only if there exists $x^* \in \partial\psi(\hat{x}(b))$ s.t. $(x^*, \hat{x}(b)) = \inf\{(x^*, y); y \in R(b)\}$.*

Proof. Let $\phi : C(T, H) \rightarrow \mathbb{R}$ be defined by $\phi(x) = (\psi \circ e_b)(x) = \psi(x(b))$, $e_b(\cdot)$ being the evaluation at b map. From proposition 5.7, p. 27, of Ekeland–Temam [9], we know that $\partial\phi(x) = e_b^* \partial\psi(x(b))$. Rewrite (O_2) as $\inf\{\phi(x); x \in Q(x_0)\}$ and apply theorem 4.1. So there exists $z^* \in \partial\phi(\hat{x}) = e_b^* \partial\psi(\hat{x}(b))$ s.t. $((z^*, \hat{x})) = \min\{((z^*, x)); x \in Q(x_0)\}$. But $z^* =$

$e_b^* x^*, x^* \in \partial\psi(\hat{x}(b))$. Hence we have:

$$\begin{aligned} ((e_b^* x^*, \hat{x})) &= (x^*, \hat{x}(b)) = \min \{((e_b^* x^*, x)): x \in Q(x_0)\} \\ &= \min \{(x^*, x(b)): x \in Q(x_0)\} \\ &= \min \{(x^*, y): y \in R(b)\}. \end{aligned}$$

Q.E.D.

5. Controllability

In this section we examine some controllability questions for the systems considered in this paper.

Let $V_0, V_b \subseteq H$ be nonempty. We say that (*) is " $V_0 V_b$ -approximately controllable," if given $\varepsilon > 0$ there exists $x_0^\varepsilon \in V_0$ and admissible control $u^\varepsilon(\cdot) \in S_U^1$ s.t. if $x^\varepsilon(\cdot) \in C(T, H)$ is the corresponding trajectory emanating from x_0^ε , we have $d(x^\varepsilon(b), V_b) < \varepsilon$.

Theorem 5.1. *If the hypotheses of theorem 3.1 hold for every $x_0 \in V_0$, and system (*) is $V_0 V_b$ -approximately controllable, then*

$$\sup_{x_0 \in V_0} \min_{|x^*| \leq 1} \sigma(x^*, R(x_0, b) - V_1) \geq 0, \text{ where } R(x_0 b) = \{x(b): x(\cdot) \in Q(x_0)\}.$$

Proof. From the definition of $V_0 V_b$ -approximate controllability, we know that there exist $\{x_0^n\}_{n \geq 1} \subseteq V_0$ s.t.

$$R(x_0^n, b) \cap (V_1 + \hat{B}_{1/n}) \neq \emptyset.$$

So for every $x^* \in H, |x^*| \leq 1$, we have

$$\sigma(x^*, R(x_0^n, b)) \geq \inf_{v \in V_1} (x^*, v) - 1/n \Rightarrow \sigma(x^*, R(x_0^n, b) - V_1) \geq -1/n.$$

Let $n \rightarrow \infty$. We get $\sup_{x_0 \in V_0} \min_{|x^*| \leq 1} \sigma(x^*, R(x_0, b) - V_1) \geq 0$.

Q.E.D.

When the system is linear the above inequality condition is also sufficient for $V_0 V_b$ -approximate controllability of the system.

So consider the following system.

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t)x(t) = B(t)u(t) \text{ a.e. on } T \\ x(0) = x_0, \\ u \in U \subseteq L^2(Z) \end{array} \right\} \quad (***)$$

Theorem 5.2. *If hypotheses $H(A), H(U), H_c$ hold, $B \in L^\infty(T, \mathcal{L}(Z, H)), V_0 = \{x_0\}$ and $\min_{|x^*| \leq 1} \sigma(x^*, R(x_0, b) - V_1) > 0$ then (***) is $V_0 V_b$ -approximately controllable.*

closed, while by hypothesis the same is true for V_b . Also $\overline{R(x_0, b)}$ is w -compact (see theorem 3.2). Thus we can apply the strong separation theorem and get $x^* \in H \setminus \{0\}$, $|x^*| \leq 1$ and $\delta > 0$ s.t.

$$\sigma(x^*, R(x_0, b)) < -\delta + \min_{v \in V_1} \sigma(x^*, v) \Rightarrow \sigma(x^*, R(x_0, b) - V_1) < -\delta$$

a contradiction to our hypothesis. So (***) is $V_0 V_b$ -approximately controllable.

Q.E.D.

When $V_b = H$, then we say that (***) is "approximately controllable" from $V_0 = \{x_0\}$ in time b , if $\overline{R(x_0, b)} = H$.

Let $B \in L^\infty(T, \mathcal{L}(Z, X^*))$ and define $\hat{B} \in \mathcal{L}(L^2(Z), L^2(X^*))$ by setting $(\hat{B}u)(\cdot) = B(\cdot)u(\cdot)$. Also we need the following weaker form of $H(A)$. $H(A)_1: A: T \times X \rightarrow X^*$ is an operator s.t. $A(t)(\cdot)$ is linear, continuous, strongly monotone and $\|A(\cdot)\|_{\mathcal{L}(X, X^*)} \in L^1_+$.

Theorem 5.3. *If $H(A)_1$ holds and $\hat{B}(U)$ is dense in $L^2(X^*)$, then (***) is approximately controllable from x_0 in time b .*

Proof. Without any loss of generality assume that $x_0 = 0$. Then consider the adjoint equation $\dot{v}(t) - A^*(t)v(t) = 0$ a.e. on T , $v(b) = h \in H$. From Lions [14] we know that this has a unique solution $v(\cdot) \in W(T) \subseteq L^2(X) \cap C(T, H)$. Therefore for every trajectory $x(\cdot)$ of (***) we have:

$$\int_0^b \langle \dot{x}(t), v(t) \rangle dt + \int_0^b \langle A(t)x(t), v(t) \rangle dt = \int_0^b \langle B(t)u(t), v(t) \rangle dt.$$

Since $v(\cdot), x(\cdot) \in W(T)$, we can integrate by parts and get that $\langle x(b), h \rangle = \int_0^b \langle B(t)u(t), v(t) \rangle dt = (\hat{B}u, v)_{L^2(X^*), L^2(X)} \Rightarrow h \in R(b, x_0)^\perp$ if and only if $(\hat{B}u, v)_{L^2(X^*), L^2(X)} = 0$ for all $u \in U$.

But $\hat{B}(U)$ is dense in $L^2(X^*)$. So $v = 0$ on $T \Rightarrow h = v(b) = 0 \Rightarrow R(b, x_0)^\perp = \{0\} \Rightarrow \overline{R(b, x_0)} = H \Rightarrow$ (***) is indeed approximately controllable.

Q.E.D.

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$\tilde{K}_3'$ – A new triangulation of R^n

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Abstract. This paper introduces a new triangulation of R^n . The triangulation is to be applied for solving system of nonlinear equation through a fixed point technique.

Keywords. Triangulation; grid size; fixed point algorithm.

1. Introduction

Since 1967 when Scarf [6] initiated a new algorithm for the computation of fixed point of a given mapping through his concept of primitive set, there have been many attempts to improve it (Khun [4], Todd [9], Eaves [1] and Eaves and Saigal [3]). Basically these algorithms require triangulation, labelling of vertices of triangulation and a particular starting point called pivot point. Todd [10] developed a new triangulation for $R^n \times [0, 1]$ and $S^n \times [0, 1]$. Eaves [2] and Eaves and Saigal [3] introduced the triangulations for $S^n \times [1, \infty]$ or $R^n \times [1, \infty]$ which decompose those spaces into finer meshes thereby generating an infinite path which approximates f better as deformation tends to infinity. The efficiency of a fixed point algorithm is sensitive to the triangulation used. Hence, the continuous refinement in the triangulation provides the improved computational technique for the fixed point algorithm to calculate the fixed point of a mapping to a greater accuracy.

2. Notation and definition

As usual R denotes the real line, Z the set of integers, E the set of even integers and O the set of odd integers. All the n -dimensional spaces have coordinates 1 through n , while the $(n+1)$ -dimensional spaces have coordinates 0 (zero) through n . Similarly, the $n(n+1)$ rows or columns of the matrix are indexed 1 (one) through n (0 through n). Π and Π_{n+1} denote the group of permutation on $\{1, 2, \dots, n\}$ and $\{0, 1, 2, \dots, n\}$ respectively. The unit vectors in R^n and R^{n+1} are denoted by $\{e^1, \dots, e^n\}$ and $\{f^0, \dots, f^n\}$ respectively.

The standard n -dimensional closed simplex S^n is the convex hull of f^0 through f^n . A K -dimensional simplex (or K simplex) σ is the relative interior of the convex hull of $(K+1)$ affinely independent points $y^0, \dots, y^K$ called its vertices. We denote the simplex by $\sigma = (y^0, \dots, y^K)$. A simplex τ is a face of σ iff its vertices are a subset of the vertices of σ and the closure of $(k-1)$ -dimensional face of K -simplex σ is called a

3. Preliminaries

3.1 Basically triangulation can be classified into two forms (details can be found in lecture notes of Todd [9]). One form is regular triangulation and the other triangulation with continuous refinement of grid size. The former consists of K , H , J_1 and K_1 developed by Khan [9], Merrill, Tucker and Freudenthal (cf. [9] for details) and the latter consists of K_3 due to Eaves and a regular [3] and J_3 due to Todd [9]. In between there are J_1 , J_2 , K_2 types of triangulation developed by Todd [10] and Eaves [2]. J_3 is the refinement of J_1 and K_3 the refinement of K_2 .

3.2 We give below the summary of earlier triangulations.

The triangulation K_1 of R^n . Let $K_1^0 = Z^n = \{y \in R^n \mid y^i \in Z \text{ for each } i \in N\}$. If $y^0 \in K_1^0$ and Π is a permutation of N , then $K_1(y^0, \Pi)$ denotes n -simplex $(y^0, \dots, y^n)$ where $y^i = y^{i-1} + u^{\Pi(i)}$ for each $i \in N$.

Finally, let K_1 be the collection of all such $K_1(y^0, \Pi)$. Whenever we set $K_1(y^0, \Pi) = (y^0, \dots, y^n)$, we suppose the y^i 's ordered as above.

The triangulation J_1 of R^n . Let $J_1^0 = Z^n$. Denoting members of J_1^0 (to be called "central vertices") by J_1^{0c} is the set $\{y \in J_1^0 \mid y_i \text{ is odd for each } i \in N\}$. If $y^0 \in J_1^{0c}$, Π is a permutation of N , and $s \in R^n$ is a sign vector (each s_i is ± 1), then denote by $j_1(y^0, \Pi, s)$ the n -simplex $(y^0, \dots, y^n)$ where $y^i = y^{i-1} + s_{\Pi(i)} u^{\Pi(i)}$ for each $i \in N$. Let J_1 be the collection of all such $j_1(y^0, \Pi, s)$.

The triangulation K_2 of S^n . Let Q denote $(n+1) \times n$ matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ +1 & -1 & \cdots & 0 \\ 0 & +1 & \cdots & \vdots \\ 0 & 0 & \vdots & -1 \\ \vdots & \vdots & \cdots & +1 \end{pmatrix}$$

and q^j its j th column for $j \in N$. Let $K_2^0(m) = \{y \in S^n \mid my_i \in Z\}$ for each $i \in N_0$. If $y^0 \in K_2^0(m)$ and Π is a permutation of N , let $\sigma = (y^0, \dots, y^n)$ where $y^i = y^{i-1} + m^{-1} q^{\Pi(i)}$ for each $i \in N$. If $\sigma \subseteq S^n$, we write $\sigma = k_2(y^0, \Pi)$ (m is implicit). Finally, let $K_2(m)$ be the collection of all such $k_2(y^0, \Pi)$.

The triangulation J_2 of S^n . Let $J_2^0(m) = K_2^0(m)$ and $J_2^{0c}(m) = \{y \in J_2^0(m) \mid my_i \text{ is even, } 1 \leq i < n \text{ and } my_n \text{ is odd}\}$. If $y^0 \in J_2^{0c}(m)$, Π a permutation of N , and $s \in R^n$ a sign vector, then $\sigma = (y^0, \dots, y^n)$ where $y^i = y^{i-1} + m^{-1} s_{\Pi(i)} q^{\Pi(i)}$ for each $i \in N$. If $\sigma \subseteq S^n$, then we write $\sigma = j_2(y^0, \Pi, s)$. Finally, let $J_2(m)$ be the collection of all such $j_2(y^0, \Pi, s)$.

3.3 We will discuss K'_3 and J_3 with continuous refinement of grid size due to Todd [10]. K'_3 is a modification of K_3 which triangulates $[0, 1] \times R^n$. Then the $(n+1)$ -dimensional simplex σ of K'_3 is given by $(y^{-1}, \dots, y^n)$ and pair (y, Π) $K'_3 \times \Pi_{n+1}$ with $\Pi(j) = 0$ and $\mu l(y) = 0$ if $\Pi^{-1}(l) > j$ is called admissible where $K'_3 0$ is the set of vertices given by $\{y \in R^{n+1} : y_0 = 2^{-k}, 0 \leq k \in Z, y_0/y_i \in Z \text{ for } 1 \leq i \leq n\}$ and $\mu: K'_3 0 \rightarrow B^{n+1}$ is

$$y^i = y^{i-1} + y_0 f^{\Pi(i)}, \text{ where } 0 \leq i < j$$

$$y^j = y^{j-1} - \sum_{i=0} \mu \Pi(i) y_0 f^{\Pi(i)} + y_0 f^0 - y_0 f^{\Pi(K)}, K=j+1$$

$$y^k = y^{k-1} + 2y_0 f^{\Pi(K)}, j < k \leq n.$$

We denote σ by $K(y, \Pi)$. $\tilde{J}_3$ is the triangulation derived for J_3 by Todd [10] by taking $\{v_1, v_2, \dots, v_n\}$ as set of unit vectors in R^n and $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1}\}$ as the set of unit vectors in R^{n+1} . Suppose $\bar{y} \in Z^n \times [1]$ have all $\bar{y}_i$'s odd let Π be the permutation of $\{1, 2, \dots, n+1\}$ with $\Pi(j) = n+1$. Let $\bar{s} \in R^n \times [-1]$ be the sign vector such that for $j < k \leq n+1$, $\bar{s}_{\Pi(k)}$ is -1 ($+1$) if

$$\bar{y}_{\Pi(i)} \text{ is } 1(3) \bmod 4.$$

Then $\tilde{J}_3(y, \Pi, \bar{s})$ denotes simplex $(\bar{y}^0, \dots, \bar{y}^{n+1})$ where

$$\bar{y}^0 = \bar{y}$$

$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}_{\Pi(i)} \bar{U}^{\Pi(i)}, \quad 1 \leq i \leq j$$

$$\bar{y}^j = \bar{y}^{j-1} - \sum_{K=j+1}^{n+1} \bar{s}_{\Pi(K)} \bar{U}^{\Pi(K)} + \bar{s}_{n+1} \bar{U}^{n+1}$$

$$\bar{y}^k = \bar{y}^{k-1} + 2\bar{s}_{\Pi(K)}, \quad j < k \leq n+1$$

$\tilde{J}$ is the set of all such $\tilde{J}_3(\bar{y}, \Pi, \bar{s})$.

4. Main result

We define new triangulation $\tilde{K}'_3$ as follows:

$\tilde{K}'_3 0$ is the set of vertices given by

$$\{\bar{y} \in R^{n+1} : \bar{y}_0 = 2^{-k}, \quad 0 \leq k \in Z$$

$$\bar{y}_i / \bar{y}_0 \in Z, \quad \text{for } 1 \leq i \leq n\} \text{ and}$$

$\mu: \tilde{K}'_3 \rightarrow B^{n+1}$ is defined by

$$\mu_i(\bar{y}) = 1 - \bar{y}_i / \bar{y}_0 \bmod 2.$$

The $(n+1)$ -dimensional simplex $\tilde{K}'_3(\bar{y}, \Pi)$ denotes the simplex $(\bar{y}^0, \dots, \bar{y}^{n+1})$, where

$$\bar{y}^0 = \bar{y}$$

$$\bar{y}^i = \bar{y}^{i+1} + \bar{y}_0 f^{\Pi(i)}, \quad 0 \leq i < j$$

$$\bar{y}^j = \bar{y}^{j+1} - \sum \mu_{\Pi(i)} \bar{y}_0 f^{\Pi(i)} + \bar{y}_0 f_0 - \sum \bar{y}_0 f^{\Pi(i)} + \bar{y}_0 f^{n+1}$$

$$\bar{y}^k = \bar{y}^{k+1} + 2\bar{y}_0 f^{\Pi(k)}, \quad j < k \leq n+1$$

$\tilde{K}'_3$ is the set of such $\tilde{K}'_3(\bar{y}, \Pi)$ and Π is the permutation on $\{1, 2, \dots, n+1\}$.

4.1 Theorem. $\tilde{K}'_3$ is the triangulation of $[0, 1] \times R^n$.

Proof. To prove $\tilde{K}'_3$ triangulation of $[0, 1] \times R^n$ we will show that simplices of $\tilde{K}'_3$ together with all these faces cover $[0, 1] \times R^n$ and none of their faces intersect.

As every neighbourhood of point of $]0, 1[\times R^n$ meets finite number of simplices, it is sufficient to prove the following two conditions:

- (1) For each $x \in]0, 1[\times R^n \exists \sigma \in \tilde{K}'_3$ such that
- (2) Each $x \in]0, 1[\times R^n$ lies in the unique face of some $\sigma \in \tilde{K}'_3$
- (1) Let k be such that $2^{-k} \leq x_0 < 2^{-k+1}$ and for $0 \leq i \leq n$, let $\bar{y}_i = 2^k [2^k, x_i]$ where $[a]$ denotes the largest integer not greater than a . Then $\bar{y} \in \tilde{K}'_3$ let

$$z = x - \bar{y} \geq 0, \quad \mu = \mu(\bar{y})$$

and

$$\bar{w} = z + z_0 \mu.$$

We again consider two cases:

Case I. The largest component $\bar{w}$ is at most $\bar{y}_0$ order the component of $\bar{w}$ using $\Pi \in \{1, 2, \dots, n+1\}$, we have

$$\bar{y}_0 \geq \bar{w}_{\Pi(j)} \geq \dots \bar{w}_{\Pi(n)} \geq -\bar{w}_0 \quad (1)$$

when $\Pi(j)=0$ and j is maximal then if $j < n$, $\bar{w}_0 > \bar{w}_l$ for $\Pi^{-1}(l) > j$ $Z_1 + Z_0 \mu l < Z_0$ since $Z_0 \geq 0$ we have $\mu l = 0 \Rightarrow (\bar{y}, \Pi)$ is admissible. Denoting the terms in (1) by

$$\alpha_0 = \bar{y}_0$$

$$\alpha_0, \dots, \alpha_{n+1} = -\bar{w}_0$$

define

$$\beta_i = (\alpha_i - \alpha_{i+1}) / \bar{y}_0, \text{ for } 0 \leq i < j$$

and

$$\beta_k = (\alpha_{k-1} - \alpha_k) / 2\bar{y}_0 \text{ for } j \leq k \leq n+1.$$

Then

$$\beta \geq 0 \quad \text{and} \quad \sum_{i=0}^{n+1} \beta_i = 1$$

and

$$x = \sum_{i=0}^{n+1} \beta_i \bar{y}^i$$

where

$$\{\bar{y}^0, \dots, \bar{y}^{n+1}\} \text{ are given by (1) thus } x \in \bar{\sigma} = K(\bar{y}, \Pi) \in \tilde{K}'_3.$$

Case II. $\bar{w}_i > y_0$ for $i \in j \leq \{1, 2, \dots, n+1\}$. Since $z_i \leq y$ for $1 \leq i \leq n$, we must have $\mu_i = 1$ for $i \in j$ increase the components of corresponding to $i \in j$ by $\bar{y}_0$ and again let $\mu = \mu(\bar{y})$, $z = x - \bar{y}$ and $w = z + z_0 \mu$. Here μ_i and $\bar{w}_i$ remain unchanged for $i \notin j$ and for $i \in j$ μ_i becomes 0 and $\bar{w}_i$ is reduced by x_0 implying that x_0 lies between 0 and

β is obtained by showing that $x \in \bar{\sigma}$ for some $\bar{\sigma} = K(\bar{y}, \Pi) \in \bar{K}'_3$. Thus x is in a face of some simplex $\bar{K}'_3$.

(II) By condition (i) we can assume that $x \in \bar{\sigma}$ for $\sigma = (\bar{y}^0, \dots, \bar{y}^{n+1}) = K(\bar{y}, \Pi)$ let

$$x = \sum_{l=0}^{n+1} \beta_l \bar{y}^l \text{ with } \beta \geq 0 \text{ and } \sum_{l=0}^{n+1} \beta_l = 1$$

then x lies in the face of whose vertices are those $\bar{y}^l$ with $l \in L = \{l: \beta_l > 0\}$. This face is called carrier of x in $\bar{\sigma}$. It is sufficient to show that $\{\bar{y}^l: l \in L\}$ is uniquely determined from x independent of $\bar{y}$ and Π . Define

$$\alpha = \alpha(x) \text{ by } \alpha_i = \bar{y}_0 \sum_{l=0}^n \beta_l \text{ for } 0 \leq i \leq j$$

and

$$\alpha_k = \bar{y}_0 \left(\sum_{l=j}^{k-1} \beta_l + \sum_{l=k}^{n+1} \beta_l \right) \text{ for } j < k \leq n+1$$

$$\Rightarrow \alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_{n+1} \text{ with } \alpha_{n+1} = -\alpha_j$$

and $\alpha_i > \alpha_{i+1}$ iff $\rho_i = 0$. These α 's are in fact the terms of (i) corresponding to x . Again define $x' \in R^{n+1}$ by

$$x' \Pi(l) = \bar{y} \Pi(l) - \mu \Pi(l) \alpha_j + \alpha_i, \quad 0 \leq l \leq n.$$

Then α 's are precisely the terms of (1) corresponding to x again it can be easily shown that

$$x' = \sum_{l=1}^{n+1} \beta'_l \bar{y}^l \Rightarrow \beta' = \beta \text{ and } x' = x.$$

Similarly, we can show that for any $z \in \bar{\sigma}$ we can obtain $\alpha(z)$ and for $l \in L$ iff the position of strict inequality in the element of $\alpha(y^l)$ also corresponds to strict inequality in $\alpha(x)$. We distinguish two cases:

Case 1. When x_0 is not power of $\frac{1}{2}$. We claim that for $l \in L \exists$ unique y^l such that we have $\alpha(y^l)$. let

$$\gamma \in]\bar{y}_0 - x_0, \bar{y}_0]$$

denote $z = z(\gamma)$ for γ lies each non-trivial interval between adjacent coordinate of $\alpha(\gamma)$ if $\gamma > x_0 - y_0$, we have for any z

$$Z_i = \begin{cases} x_i & \text{if } x_i/\bar{y}_0 \in E, \\ x_i + x_0 - \bar{y}_0 & \text{if } (x_i + x_0 - \bar{y}_0)/\bar{y}_0 \in 0. \end{cases}$$

To determine $\bar{y}_i$ and $\bar{w}_i$ from x , we apply

$$Z_i = \begin{cases} \bar{y}_i + \bar{y}_0 & \text{for } \bar{w}_i \geq \gamma \\ \bar{y}_i & \text{for } \bar{w}_i < \gamma \text{ (this case includes } i = 0). \end{cases}$$

If $\gamma \leq x - \bar{y}_0$, then z can be obtained as follows:

$$Z_i = \begin{cases} x_i & \text{if } x_i/\bar{y}_0 \in E, \\ x_i + x_0 - 2\bar{y}_0, & \text{if } (x_i + x_0 - \bar{y}_0)/\bar{y}_0 \in 0. \end{cases}$$

To determine $\bar{y}_i$ and $\bar{w}_i$ from we apply for when

$$Z = \begin{cases} \bar{y}_i + \bar{y}_0 & \text{for } \bar{w}_i \geq \gamma \text{ (this case includes } i=0) \\ \bar{y}_i - \bar{y}_0 & \text{for } \bar{w}_i < \gamma. \end{cases}$$

Thus showing that vertices of the carrier of $x \in \bar{\sigma}$ can be obtained from x alone.

Case 2. When x is a power of $\frac{1}{2}$ since each y'_0 is a power of $\frac{1}{2}$ and only two adjacent powers may appear $y'_0 = x \forall l \in L$ showing that all vertices of σ with the same 0th coordinates for facing $\bar{K}'_3$ thereby proving the result in this case as well.

5. Comparative analysis

We have introduced a new triangulation known as $\bar{K}'_3$ which is a modification of K_3 and K'_3 developed by Eaves and Saigal [8] and Todd [9] respectively. K'_3 is purely an extension of K_3 by including the boundary of rectangle used in K'_3 on the right hand side. Todd [9] has described various theoretical measures of the efficiency of triangulation for computing fixed point and determine their values for all the triangulations described by him on the basis of their measure. He found that J_1 and K_1 are roughly comparable and both are superior to H . J_1 is less sensitive to different direction than K . In horizontal direction J_3 and K'_3 are again comparable, but in the vertical direction J_3 appears to be superior.

Similarly $\bar{K}'_3$ is comparable with J_3 in horizontal direction and very much equivalent to K'_3 as both J'_3 and K'_3 have been coded in Fortran for the purpose of computation. So $\bar{K}'_3$ can be coded in a similar way for the computation of non-linear problems.

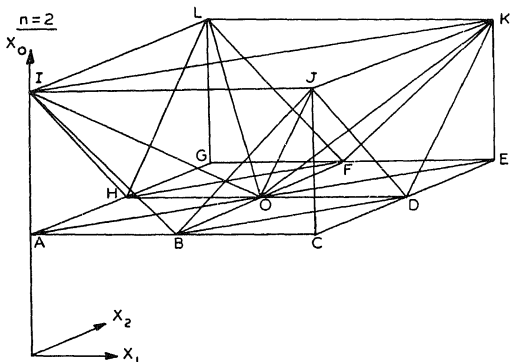
Besides, this new triangulation is superior because its algorithms will take a few steps to calculate accurately fixed point for any mapping f . Any infinite path generated to the infinite sequence gives distinct and complete simplices of homotopy h defined for the mapping f . In summary.

Step 1. Let σ_1 be the unique $(n+1)$ -simplex of G with τ_0 as facet where G denotes a triangulation with continuous refinement of grid size. Let y_0^+ with the vertex σ_1 outside $R_{(0)}^n$ be a linear order system showing τ_0 as a complete set $m \leftarrow 1$.

Step 2. Define $l: [0, 1] \times R^n \rightarrow R^n$ by $l(x) = h(x) - P(x)$ and calculate $l(\bar{y}^+)$. Introduce the column $(l(\bar{y}^+))$ into the current basis of linear order system. Assume that the column $(l(1/(y^+)))$ drops from the basis with $\bar{y}'$ a vertex of σ_m .

Step 3. Let τ_m be the complete facet of σ_m opposite $\bar{y}'$. If desired, calculate the fixed point of homotopy h in τ_m and ferminal if it is sufficiently accurate. Let σ_{m+1} be the simple of G sharing the facet τ_m with σ_m let y be the new vertex, set $m \leftarrow m+1$ and return to step 1.

Generally the fixed point of homotopy h in $\bar{\tau}_m$ is calculated only when $\tau_m \subset R^n$



This is the triangulation by $\tilde{k}_3$ of $[\frac{1}{2}, 1] \times [0, 1]^2$. A list of the simplices follows. (Note that without the stipulation of admissibility, other simplices intersecting these (such as AIJK, ABIL, BCJK) would appear).

$\tilde{k}_3(y, \Pi)$	y	Π	$\tilde{k}_3(y, \Pi)$	y	Π
(ABOI)	A	(1, 2, 0)	(AHOI)	A	(2, 1, 0)
(BODJ)	B	(2, 1, 0)	(BOIJ)	B	(2, 0, 1)
(BCDJ)	B	(1, 2, 0)	(ODJK)	O	(1, 0, 2)
(ODEK)	O	(1, 2, 0)	(OFLK)	O	(2, 0, 1)
(OFEK)	O	(2, 1, 0)	(HOIL)	H	(1, 0, 2)
(HOFL)	H	(1, 2, 0)	(OIJK)	O	(0, 1, 2)
(HGFL)	H	(2, 1, 0)	(OILK)	O	(0, 2, 1)

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On shock dynamics

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Abstract. This is in continuation of our paper 'On the propagation of a multi-dimensional shock of arbitrary strength' published earlier in this journal (Srinivasan and Prasad [9]). We had shown in our paper that Whitham's shock dynamics, based on intuitive arguments cannot be relied on for flows other than those involving weak shocks and that too with uniform flow behind the shock. Whitham [12] refers to this as misinterpretation of his approximation and claims that his theory is not only correct but also provides a natural closure of the open system of the equations of Maslov [3]. The main aim of this note is to refute Whitham's claim with the help of an example and a numerical integration of a problem in gasdynamics.

Keywords. Shock propagation; non-linear waves; compressible flow.

1. Compatibility conditions on a shock manifold

Significance of a characteristic manifold of a hyperbolic system of first order partial differential equations in $m + 1$ independent variables is well known and lies in the fact that one can derive from the equations a compatibility condition which contains only m independent interior derivatives in the manifold. One of the interior derivatives can be chosen in the direction of the bicharacteristic curve (or along a ray). It is also true that no surface other than the characteristic surface has the property that one can derive from the equations a compatibility condition on that surface containing only the interior derivatives. Therefore, if we take a shock manifold in $m + 1$ dimensional space of all independent variables, the differential equations can be used to derive on it a relation which contains m interior derivatives in the shock manifold (one of which may be chosen to be the direction given by the shock ray) and one exterior derivative. In our paper [9] we derived one such compatibility condition for gasdynamic equations with special aim that all the derivatives appearing in it should be that of the excess of the density behind the shock (extension of ρ_1) over that ahead of the shock (ρ_0) divided by ρ_0 i.e. $(\rho_1 - \rho_0)/\rho_0 = \mu$, say, a measure of the shock strength. For two space dimensions, we have shown that

$$\frac{\partial \mu}{\partial s_1} = B_1(\mu) \frac{\partial \Theta}{\partial \eta} + B_2(\mu) \frac{\partial \mu}{\partial N}, \quad (1)$$

and

$$\frac{\partial \Theta}{\partial s_2} = - \frac{\gamma + 1}{2(1 + \mu)\{2 + \mu(1 - \gamma)\}} \frac{\partial \mu}{\partial n}, \quad (2)$$

where $(\cos \Theta, \sin \Theta)$ is the unit normal to shock surface at time t , $\partial/\partial s_1$ is the spatial rate of change in (x_1, x_2, t) -space along a shock ray, $\partial/\partial \eta$ and $\partial/\partial N$ are the tangential and normal derivatives for the shock surface at a fixed time, and B_1 and B_2 are functions of μ and the ratio γ of the specific heats (Srinivasan and Prasad [10]). The derivatives $\partial/\partial s_1$ and $\partial/\partial \eta$ are interior derivatives in the shock manifold in (x_1, x_2, t) -space and $\partial/\partial N$ is an exterior derivative.

Equation (1) is just one of the infinite sequence of compatibility conditions, which can be derived on the shock manifold by Grinfeld and Maslov's methods ([2], [3]). It throws a deep insight into a new structure of the gasdynamic equations: the rate of change of the shock amplitude depends not only on the distribution of the shock strength on the shock surface but also on the gradient of the flow in the direction of the normal to the shock. This is an exact result. At any time, this gradient can be made to have an arbitrary value by suitably prescribing the initial data influencing the flow behind the shock and hence the last term in (1) represents the effect of the waves which catch up with the shock from behind.

Whitham's theory [11] 'shock dynamics' is based on a number of assumptions (such as rays are particle paths, see page 277 [11]), the two main assumptions are

Assumption 1. Evolution of the shock strength on the shock front depends only on the initial position of the shock front and the distribution of the physical variables (and not on their derivatives) on the shock front.

Assumption 2. The rate of change of the shock strength along a ray is given by the characteristic rule.

Assumption 1, which has never been stated explicitly by Whitham, was not realised and pointed out earlier to our work [4], [8]. This assumption is mathematically equivalent to having a compatibility condition on the shock manifold with only first order interior derivatives appearing in it. This assumption leads to Whitham's basic equations (equations (8.59) and (8.60) of [11]) and amounts to having only terms containing the derivatives $\partial/\partial s_1$ and $\partial/\partial \eta$ in (1). Assumption 2, giving a quantitative relation between shock strength and the ray tube area, is independent of assumption 1. The exact compatibility condition (1) shows that assumption 1 is not correct. To deduce his equation (8.59) in [11] from (1), Whitham assumes without any justification

$$\frac{\partial \mu}{\partial N} = a(\mu) \frac{\partial \mu}{\partial s_1}. \quad (3)$$

Since

$$\frac{\partial}{\partial s_1} = \frac{1}{C} \left(\frac{\partial}{\partial t} + C \cos \Theta \frac{\partial}{\partial x_1} + C \sin \Theta \frac{\partial}{\partial x_2} \right),$$

$\partial/\partial N = \cos \Theta \partial/\partial x_1 + \sin \Theta \partial/\partial x_2$ and μ has three independent arguments t, s and N (or t, x_1, x_2) (independent because at any time t the function μ i.e. the state behind the shock can be arbitrarily prescribed), there is no justification for the assumption (3). Assumption 2 is also not correct and it is well known that it gives reasonable

to give a mathematical justification for the assumption 2 as evident from his remark (page 272, [11]): 'When the quick derivation of the characteristic rule occurred to me, I hoped also that a full analysis of the approximation could be based directly on the original fluid dynamic equations. So far this has not been completed!' Of course, this cannot be completed because the characteristic method is not a correct approximation for a majority of problems as shown by the example in the next section.

2. An example

Example. Consider an one-dimensional single conservation law

$$u_t + (\tfrac{1}{2}u^2)_x = 0 \quad (4)$$

for which the shock velocity S is given in terms of the state u_l on the left and u_r on the right by

$$S = \tfrac{1}{2}(u_l + u_r). \quad (5)$$

We assume that the restriction of the solution to a region immediately on the left of the shock is smooth, then u satisfies $u_t + uu_x = 0$ or $u_t + \tfrac{1}{2}(u + u_r)u_x + \tfrac{1}{2}(u - u_r)u_x = 0$. Taking the limit as we approach the shock from the region on the left, we deduce the compatibility condition along the shock path in (x, t) -plane as

$$\frac{du_l}{dt} + \tfrac{1}{2}(u_l - u_r)(u_x)_l = 0 \quad \text{along} \quad \frac{dx}{dt} = \tfrac{1}{2}(u_l + u_r). \quad (6)$$

Unlike the compatibility condition $du/dt = 0$ on the characteristic curve along $dx/dt = u$, the condition (6) contains an exterior derivative with respect to the shock curve in addition to an interior derivative $d/dt = \partial/\partial t + S(\partial/\partial x)$. The exterior derivative term is $\tfrac{1}{2}(u_l - u_r)(u_x)_l$ which corresponds to the term $B_2(\mu)(\partial/\partial N)$ in (1).

While criticising our work, Whitham makes another (wrong) assumption, namely by the relation (3), which for this example, becomes

$$(u_x)_l = a(u_l) \left(\frac{\partial u_l}{\partial t} + S \frac{\partial u_l}{\partial x} \right).$$

Using it in (5), we get

$$\{1 + \tfrac{1}{2}a(u_l)(u_l - u_r)\} \frac{du_l}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = \tfrac{1}{2}(u_l + u_r),$$

a compatibility condition containing only the interior derivatives. This implies (unless $1 + \tfrac{1}{2}a(u_l)(u_l - u_r) = 0$) that u_l is constant along the shock path irrespective of the state behind the shock. Thus, this assumption (3) of Whitham leads to an absurd result. This is also shown by considering the following initial value problem (Ramanathan [7]) for the conservation law (4):

$$u(x, 0) = \begin{cases} 0, & x < -\eta \\ \frac{x + \eta}{\eta + 1}, & -\eta \leq x < 1 \\ 0, & x \geq 1 \end{cases} \quad (7)$$

with $\eta > -1$. Note that the initial data contains a single shock at $x = 1$. The shock position $X(t)$ at any time t is given by

$$X(t) = -\eta + (1 + \eta)^{1/2}(1 + \eta + t)^{1/2}. \quad (8)$$

As $\eta \rightarrow \infty$, $u(x, 0)$ gives a uniform state behind the shock and in this case the shock moves with a constant velocity $\frac{1}{2}$. For all other values of η , $-1 < \eta < \infty$, the shock velocity and shock strength are respectively

$$S \equiv \frac{dX}{dt} = \frac{1}{2} \left(\frac{1 + \eta}{1 + \eta + t} \right)^{1/2} < \frac{1}{2} \quad \text{and} \quad u_1 = \left(\frac{\eta + 1}{1 + \eta + t} \right)^{1/2} \quad (9)$$

which can differ by any amount (in per cent error) from the respective values $1 + \frac{1}{2}t$ and $\perp$ given by Whitham's Characteristic rule.

3. An example from gasdynamics

We have also used (1) and (2) to develop an algorithm to compute successive positions of a curved shock, when the shock is weak (Ramanathan [7], Ravindran and Prasad [8]). We present here the results of a computation using this algorithm and Whitham's theory from the Ph.D. thesis of Ramanathan for waves produced by a cuncave piston. The figure gives successive positions of a shock front according to

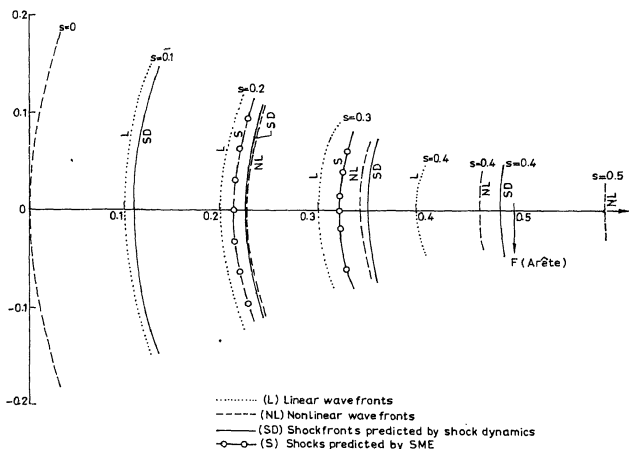


Figure 1. Initial parabola $y^2 = x$ ($w = 0.1 \exp(-0.25|\delta|)$). At $s = 0.1$ —NL, SD, S lie close to each other. At $s = 0.2$ —SD is just behind NL and at $s = 0.3$ —SD has formed ahead of NL at

Whitham's shock dynamics (referred to SD in the figure), the shock fronts (S) as predicted by our theory, the linear wavefront (L) and the nonlinear wavefront (LS) (we have also developed a theory 'Kinematics of a nonlinear wavefront' to compute weakly nonlinear wavefronts). In this particular case, the shock predicted by Whitham's theory moves quite ahead of ours. At $t' = 0.3$ and $t' = 0.4$ it has moved even ahead of the nonlinear wavefront (NL), a result which is certainly not correct, since it contradicts a well known result 'the shock velocity of a weak shock is the mean of the characteristic velocities ahead and behind the shock'.

4. Conclusion

Equation 1 is just one of the infinite sequence of compatibility conditions, which can be derived on the shock manifold. It throws a deep insight into a new structure of the gasdynamic equations, the rate of change of the shock amplitude depends not only on the distribution of the shock strength on the shock surface but also on the gradient of the flow in the direction of the normal to the shock. This is an exact result. The term $\partial\mu/\partial N$ is zero when the flow behind the shock is taken uniform in the normal direction at each instant. Except for this very special case, $\partial\mu/\partial N \neq 0$ and, in general, would contribute to the solution. As shown in the example, the value of $\partial\mu/\partial N$ will significantly depend on the initial data and hence this term must be treated as independent of the other two terms in (1). This being so, if we wish to compare our theory with Whitham's shock dynamics we must compare the coefficients B_1 and B_3 as done in our previous paper. Even in the case when the flow behind the shock is uniform i.e. $\partial\mu/\partial N = 0$, the shock dynamics of Whitham is not fully justified. Equations (1) and (2) reduce to equations similar to the shock dynamics equations but the coefficient B_1 differs significantly from B_3 in the shock dynamics for all values of μ other than those of weak shocks (see [10]). The column for $|((B_3 - B_1)/B_1)|$ in the table shows that the error is more than 50% for $M = 2.5$, about 100% for $M \approx 3.5$ and increases to 255% for strong shocks. While criticising our work, Whitham completely ignores the significant difference present in the table for intermediate values of M .

We finally note that there is neither a mathematical proof for the derivation of Whitham's shock dynamics equations nor has any one shown that it can be derived as an approximate theory. There are only two justifications for its use: (i) it has given accurate results for certain problems (specially, some of the results of characteristic rule are amazingly good) and (ii) it is a simple and elegant method to solve a large variety of complex problems (Whitham [11]). However, there are situations (Hayes, [1]) of exact similarity solutions where the error by the characteristic rule (which forms the basis of shock dynamics) is as much as 15%. Our example in §2 shows that when $t \gg (1 + \eta)$ the characteristic rule gives an error for the shock position which is even more than 100%. Our computation of a flow problem in §3 shows that Whitham's theory gives a result which is absurd since the shock front moves ahead of the nonlinear wavefront. In mathematics, there is a very important difference between verification and proof. Verifying any number of times does not constitute a proof of a theory. A single large disagreement with the results of a theory, verified even hundred times for particular problems, immediately shows that the theory is not based on a sound foundation. In Whitham's shock dynamics and the characteristic rule, there is

exact solution. Unless the set of problems, where shock dynamics can be safely used, is identified and unless an estimation of error is found, it is not correct to use Whitham's method. Not only that, it leads to a great misunderstanding about the correct form of equations on the shock manifold. Actually, Whitham shock dynamics for a weak shock is nothing but the kinematics of a nonlinear wavefront developed by us (Ravindran and Prasad, [8]) with some changes in the coefficients of the equations.

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Corrections to some expressions in

“On the propagation of a multi-dimensional shock of arbitrary strength”

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In our paper, which we published in this journal (Vol. 94, 1985, 27–42), the algebraic expressions for the coefficients $B_1(\mu)$, $B_2(\mu)$ and $B_3(\mu)$ contain some errors. The theory and the conclusions in the paper remain unaffected. The expressions are

$$\frac{\partial}{\partial \eta} = \cos \Theta \frac{\partial}{\partial x_2} - \sin \Theta \frac{\partial}{\partial x_1} \quad (45)$$

$$B_1 = -\frac{2\mu\{2 + \mu(1 - \gamma)\}}{8 + 5\mu - 3\gamma\mu - \mu^2(1 - \gamma^2)} \quad (47)$$

$$B_2 = \frac{\mu(1 + \gamma)\{2 + \mu(1 - \gamma)\}}{\{8 + 5\mu - 3\gamma\mu - \mu^2(1 - \gamma^2)\}(1 + \mu)} \quad (48)$$

and

$$\begin{aligned} B_3 = & -2\mu(1 + \mu)\{2 + \mu(1 - \gamma)\}\{2 + \mu(\gamma + 1)\}/[8\mu(1 + \mu) \\ & + \{2 + \mu(1 + \gamma)\}\{4 + \mu(3 - \gamma)\} \\ & + \sqrt{2}\{2 + \mu(1 + \gamma)\}^{1/2}\{4 + 8\mu + \mu^2(3 - \gamma)\}]. \end{aligned} \quad (55)$$

Consequently, the table 1 of the paper should be corrected as:

Table 1. Comparison between Whitham's coefficient and the corresponding coefficient obtained by us for $\gamma = 1.4$.

μ	M	B_1	B_3	$\left \frac{B_3 - B_1}{B_1} \right $	B_2
0.0	1	0	0	—	0
0.1	1.06	-0.0495	-0.0474	0.0424	-0.0529
0.2	1.12	-0.0937	-0.0905	0.0342	-0.0937
0.5	1.3	-0.2083	-0.2027	0.0269	-0.1667
0.75	1.43	-0.2790	-0.2812	0.0079	-0.1913
1.0	1.58	-0.3280	-0.3477	0.0601	-0.1967
1.25	1.73	-0.3571	-0.4030	0.1289	-0.1905
1.5	1.9	-0.3695	-0.4474	0.2108	-0.1770
2.0	2.24	-0.3571	-0.5039	0.4111	-0.1429
2.5	2.26	-0.3125	-0.5185	0.6592	-0.1071
3.0	3.16	-0.2521	-0.4923	0.9528	-0.0756
3.5	3.86	-0.1862	-0.4262	1.2890	-0.0496
4.0	5	-0.1205	-0.3215	1.6680	-0.0289
4.5	7.42	-0.0580	-0.1791	2.0879	-0.0008
$\mu \rightarrow 5 - 0$	$M \rightarrow \infty$	$-\frac{1}{5}(5 - \mu)$	$-0.394(5 - \mu)$	2.55	$-\frac{2}{35}(5 - \mu)$

In the above table M refers to the Mach number of the shock which is defined as the ratio of the shock velocity to the sound velocity ahead of the shock. From the table we see that for smaller values of μ ($\mu < 1$) the coefficients B_1 and B_3 agree well, while for higher values of μ , they differ significantly.

Singular pencils of quadrics and compactified Jacobians of curves

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Abstract. Let Y be an irreducible nodal hyperelliptic curve of arithmetic genus g such that its nodes are also ramification points ($\text{char} \neq 2$). To the curve Y , we associate a family of quadratic forms which is dual to a singular pencil of quadrics in $\mathbb{P}^{2g+1}$ with Segre symbol $[2 \dots 2 \ 1 \dots 1]$, where the number of 2's is equal to the number of nodes. We show that the compactified Jacobian of Y is isomorphic to the space R of $(g-1)$ dimensional linear subspaces of $\mathbb{P}^{2g+1}$ which are contained in the intersection Q of quadrics of the pencil. We also prove that (under this isomorphism) the generalized Jacobian of Y is isomorphic to the open subset of R consisting of the $(g-1)$ dimensional subspaces not passing through any singular point of Q .

Keywords. Nodal hyperelliptic curve; compactified Jacobian.

1. Introduction

The geometry of a nonsingular pencil of quadrics Q is intimately connected with some moduli spaces of vector bundles on the nonsingular hyperelliptic curve associated to Q ([2], [3] etc). A few years back Newstead [5] studied the geometry of singular pencils of quadrics in $\mathbb{P}^5$. As he has remarked, a very interesting problem will be to extend the above connection between the geometry of a pencil and vector bundles on the associated hyperelliptic curve to the singular case. We have taken here one step in this direction.

Let A and B be matrices corresponding to two nonsingular members of a pencil Q . If the equation $\det(A - xB) = 0$ does not have all roots distinct, then Q is a singular pencil. The simplest case is when this equation has at most double roots. Then one can associate to Q a singular curve Y with the following properties: (i) Y has at the most ordinary double points as singularities; (2) Y is a two sheeted covering over $\mathbb{P}^1$ with the set of singular points contained in the set of ramification points.

In this paper, we start with an irreducible curve Y with the above two properties and construct a pencil of quadrics Q in $\mathbb{P}^{2g+1}$, g being the arithmetic genus of Y (see §3). We denote by R the space of $(g-1)$ -dimensional subspaces of $\mathbb{P}^{2g+1}$ which are isotropic for all the members of the pencil Q . Then our main theorem is:

Theorem. *The compactified Jacobian of Y is isomorphic to R .*

2. Notations and Preliminaries

Let Y be an irreducible reduced curve with ordinary double points and F a torsion free coherent sheaf of rank one on Y .

DEFINITION 2.1

The degree of F ($\deg F$) is the integer $\chi(F) - \chi(\theta_Y)$ (I.4.1, [4]).

2.2. Since Y is Gorenstein, the dualising sheaf ω_Y is a locally free sheaf of rank one, of degree $2g-2$. (I.4.2, [4]).

2.3. (a) $\deg(F) < 0$ implies that $H^0(F) = 0$,

(b) $\deg(F) > 2g-2$ implies that $H^1(F) = 0$ (I.4.6[4]).

2.4. Let $\bar{J}$ denote the compactified Jacobian of Y . Tensorising by a line bundle of degree d , $\bar{J}$ can be identified with the space of torsion free coherent sheaves of rank one and degree d .

By Proposition 10.1, [6], $\bar{J} = \coprod \text{Pic } Y_i$ where $\coprod$ denotes disjoint union over all partial normalizations Y_i of Y . Thus, if Y has only one singular point, $\bar{J} = \text{Pic } Y \coprod \text{Pic } \tilde{Y}$, $\tilde{Y}$ being the normalization of Y .

Henceforth we make the following assumptions. (1) Y is hyperelliptic i.e. there exists a finite degree two map $p: Y \rightarrow \mathbb{P}^1$ (2) The unique node w_0 of Y is a ramification point of p . Let $W_0 = \{w_0, w_1, \dots, w_{2g}\}$ be the set of ramification points of Y . Let i denote the hyperelliptic involution on Y . We define an i -action on a torsion free coherent sheaf E on Y as a $\mathbb{Z}/2$ -action on E which lifts the involution i . We call E i -invariant if $E \approx i^*E$.

Lemma 2.5. (a) *There is a one-one correspondence between i -invariant line bundles of even degree (respectively odd degree) on Y and the set of partitions of $W_0 = \{w_0, w_1, \dots, w_{2g}\}$ into two subsets S, T such that $S \cap \{w_1, \dots, w_{2g}\}$ and $T \cap \{w_1, \dots, w_{2g}\}$ have even cardinality (respectively odd cardinality).*

(b) *There is a one-one correspondence between i -invariant nonlocally free torsion free sheaves of rank 1 and even degree (respectively odd degree) and partitions of $\{w_1, \dots, w_{2g}\} = W_0 - w_0$ into two subsets S', T' of odd (respectively even) cardinality.*

Proof. This lemma can be proved as in the case when Y is nonsingular, so we omit the details.

(a) The correspondence is given by (zero degree case)

$$L = \left\{ \bigotimes_{\omega \in S} \theta_Y(\omega) \otimes \theta_Y \left(-\frac{\#S}{2} \right) \mid w_0 \notin S, \right.$$

$$\left. L = \bigotimes_{\omega \in S \setminus w_0} \theta_Y(\omega) \otimes \theta_Y \left(\frac{1 - \#S}{2} \right) \otimes M \mid w_0 \in S, \text{ where } \theta_Y(d) = p^* \theta_{\mathbb{P}^1}(d) \right\}$$

and M is the unique squareroot of θ_Y which becomes trivial on pulling back to $\tilde{Y}$, the normalization of Y . The set S is characterized as the set of points w in W_0 such that i -action on the fibre of L at w is (-identity). The odd degree case can be dealt with similarly.

(b) Let $\pi: \tilde{Y} \rightarrow Y$ be the normalization map. The map $\text{Pic}(\tilde{Y}) \rightarrow T(Y) = \{\text{nonlocally}$

moreover L is i -invariant iff $\pi_* L$ is so. Thus i -invariants in $\text{Pic } \tilde{Y}$ are in bijective correspondence with i -invariants in $T(Y)$. Using this and the correspondence between i -invariants in $\text{Pic } \tilde{Y}$ and partitions of $W_0 - w_0$, we get the result. In case $\text{degree} = 0$, one has

$$L = \bigotimes_{\omega \in S'} \theta_Y(w) \otimes \theta_Y \left(\frac{-1 - \#S'}{2} \right) \otimes \pi_*(\theta_{\tilde{Y}}), \text{ for } s' \subset \{w_1, \dots, w_{2g}\}, \#S' \text{ odd.}$$

S' is the set of points w in $W_0 - w_0$ such that i -action on the fibre of $(\pi^* L / \text{torsion})$ at w is negative of the identity.

3. Quadratic form associated to a singular curve

3.1. Let Y be an irreducible curve with a single node at w_0 . Let $p: Y \rightarrow \mathbb{P}^1$ be a double cover with $w_0, w_1, \dots, w_{2g}$ the ramification points. The image of w_i in $\mathbb{P}^1$ will again be denoted by w_i , for all i . Consider the divisor $W = 2w_0 + w_1 + \dots + w_{2g}$ in $\mathbb{P}^1$. We have $\theta_W = (\theta/m_{w_0}^2) \oplus \left(\bigoplus_{i=1}^{2g} \theta/m_{w_i} \right)$. The multiplication in θ induces quadratic maps A_i on θ/m_{w_i} and A_0 on $\theta/m_{w_0}^2 \approx \theta/m_{w_0} \oplus m_{w_0}/m_{w_0}^2$ given by $A_i(X_i) = X_i^2$, $A_0(X_0, Y_0) = (X_0^2, 2X_0 Y_0)$. Identifying θ_W with k^d , $A = \bigoplus_{i=0}^{2g} A_i$ is a quadratic map from k^d to k^d ($d = 2g + 2$). Now, given an element $b = (b_0, b_{00}, b_1, \dots, b_{2g})$ of k^d , there exists a polynomial f_b of degree $2g + 1$ such that $f_b(w_i) = b_i$ for $i \geq 0$, and the derivative $f'_b(w_0) = b_{00}$. In fact

$$f_b(x) = \sum_{i \geq 0} b_i \mu_{w_i}(x) + \lambda \left(\prod_{i \geq 0} (x - w_i) \right)$$

where

$$\mu_{w_i}(x) = \prod_{j \neq i} (x - w_j) / \prod_{j \neq i} (w_i - w_j) \text{ and } \lambda = \left(b_{00} - \sum_{i \geq 0} b_i \mu'_{w_i}(w_0) / \prod_{i > 0} (w_0 - w_i) \right).$$

Then

$$f_b(x) = \sum_{i \geq 0} b_i \lambda_i(x) + \frac{1}{2} b_{00} \lambda_{00}(x),$$

where

$$\lambda_i(w_j) = \delta_{ij},$$

$\lambda_{00}(w_j) = 0$ for all i . Define $\phi_x: k^d \rightarrow k$ by $\phi_x(b) = f_b(x)$. Let $Q_x = \phi_x \circ A$. Then $\{Q_x\}_x$ is a family of quadratic forms on k^d parametrized by $\mathbb{P}^1$ with values in $\theta_p(2g + 1)$ or equivalently a quadratic form Q on the trivial bundle $\mathbb{P}^1 \times k^d$ with values in $\theta_{\mathbb{P}^1}(2g + 1)$. Tracing the above maps, one gets

$$(*) \quad Q_x(X_0, Y_0, X_1, \dots, X_{2g}) = \sum_{i=0}^{2g} \lambda_i(x) X_i^2 + \lambda_{00}(x) X_0 Y_0$$

$$\text{with } \lambda_i(w_j) = \delta_{ij}, \quad \lambda'_i(w_0) = 0, \quad \lambda_{00}(w_i) = 0 \text{ for all } i.$$

3.2. We make a few observations regarding Q . The space $(0, Y_0, 0, \dots, 0)$ is isotropic for Q_x for all x , this is the canonical one dimensional subspace corresponding to $m_{w_0}/m_{w_0}^2$. Also $Q_x = X_i^2$ $i = 0, 1, \dots, 2g$ and these are precisely the singular members

of the family. In fact (*) shows that Q is dual (adjoint) to a singular pencil of the form

$$Q_1 = \sum_{i=1}^{2g} X_i^2 + 2X_0 Y_0$$

$$Q_2 = \sum_{i=1}^{2g} a_i X_i^2 + X_0^2 + 2a_0 X_0 Y_0,$$

with $\{a_i\}$ distinct scalars. This is a pencil with Segre symbol $[11 \dots 12]$. We recall that the adjoint of a quadratic form Q is given by the adjoint of the matrix of Q .

3.3. We now give an alternative way of defining Q . We will interpret the maps A_j to have values in $(\theta_{p^1}(d-1))_{w_i}$, $i = 1, \dots, 2g$, and A_0 to have values in $(\theta_{p^1}(d-1)) \otimes \theta/m_{w_0}^2$. The evaluation map $\psi: H^0(\theta_{p^1}(d-1)) \rightarrow \theta_{p^1}(d-1) \otimes \theta_w$ can easily be checked to be an isomorphism. Let e_x be the evaluation map $H^0(\theta_{p^1}(d-1)) \rightarrow (\theta_{p^1}(d-1))_x$. Then $\phi_x = e_x \circ \psi^{-1}$; so that $e_x \circ \psi^{-1} \circ A$ gives Q_x .

3.4. *The general case.* We have restricted ourselves to the case of a single node merely for the simplicity of exposition. The construction in 3.3 works in the case of many ordinary double points too. In that case, for W , we have to take the divisor $\sum_i 2v_i + \sum_j w_j$, where v_i varies over all ordinary double points of Y and $\{v_i, w_j\}_{i,j}$ is the set of ramification points for the double cover $Y \rightarrow \mathbb{P}^1$. The quadratic form Q is then adjoint to the pencil of the form

$$Q_1 = \sum_{i=1}^n X_i^2 + \sum_{j=1}^m Z_j Y_j$$

$$Q_2 = \sum_{i=1}^n a_i X_i^2 + \sum_{j=1}^m (Z_j^2 + 2b_j Z_j Y_j)$$

where a_i, b_j are distinct scalars, m equals the number of ordinary double points; $2m+n=2g+2$, g being the arithmetic genus of Y .

3.5. In this and the next article we want to study a subvariety of the variety of g dimensional linear subspaces of k^{2g+2} which are isotropic for the singular pencil

$$Q_1 = \sum_{i=1}^{2g} X_i^2 + 2X_0 Y_0$$

$$Q_2 = \sum_{i=1}^{2g} a_i X_i^2 + X_0^2 + 2a_0 X_0 Y_0$$

where a_i 's are all distinct scalars. We have a splitting $k^{2g+2} = \bigoplus_{i=1}^{2g} C_i + C_0$ where X_i is the coordinate in C_i for $i = 1, \dots, 2g$ and X_0, Y_0 are the coordinates in C_0 . The subspace of C_0 defined by $X_0 = 0$ will be denoted by C_0^0 . For simplicity of notation we will often denote $\bigoplus_{i=1}^{2g} C_i$ by k^{2g} .

Let $\text{Grass}_r(\quad)$ denote the grassmannian of r -dimensional subspaces. Define P, P_0 by

$$P = \left\{ V_0 \in \text{Grass}_g(k^d) \mid \begin{array}{l} V_0 \text{ isotropic for the pencil} \\ V_0 = V_1 \oplus C_0^0, V_1 \subset k^{2g} \end{array} \right\}$$

$$P_0 = \left\{ V_1 \in \text{Grass}_{g-1}(k^{2g}) \mid \begin{array}{l} V_1 \text{ isotropic for the pencil} \\ \text{restricted to } k^{2g} \end{array} \right\}.$$

and

Clearly, $P \approx P_0$.

Remark. It is well known that the pencil on k^d induces a dual family on $(k^d)^*$, if a quadratic form on k^d is given by matrix B , the corresponding form on $(k^d)^*$ is given by the adjoint of B . Under the natural correspondence between $\text{Grass}_g(k^d)$ and $\text{Grass}_{g+2}(k^d)^*$ which maps a subspace $V_0 \subset k^d$ to the subspace V of linear forms on k^d vanishing on V_0 , P corresponds to the subvariety P' of $\text{Grass}_{g+2}(k^d)^*$ consisting of subspaces of the form $V = V_1 \oplus C_0^0$ (C_0^0 being the unique subspace of $(C_0)^*$ isotropic for the dual family) such that any generic member of the (dual) family restricted to V has rank 2. Also, if we identify (k^d) with $(k^d)^*$ using the isomorphism given by a generic member of the pencil, say Q_1 , then V gets identified to the orthogonal complement of V_0 with respect to Q_1 .

Lemma 3.6. If V_0 is isotropic for the pencil but V_0 does not belong to P , then

$$V'_0 + k^{2g} = k^d$$

where V'_0 denotes the orthogonal complement of V_0 with respect to a generic form of the pencil.

Proof. Since V_0 is isotropic and the only subspace of C_0 isotropic for a generic form is C_0^0 it follows that $V_0 \cap C_0 = C_0^0$ or $\{0\}$. Hence V_0 is not in P iff $V_0 \cap C_0 = \{0\}$. Taking orthogonal complements with respect to a generic form, the latter is equivalent to $V'_0 + k^{2g} = k^d$.

Remark 3.7. Lemma 3.6 says that V is not in P' iff $\dim V \cap k^{2g} = g$. For V in P' , $\dim V \cap k^{2g} = g + 1$.

4. The generalized Jacobian

With the notations of 3.1, and 3.5, the main result of the section is the following.

Theorem 1. The generalized Jacobian of Y is isomorphic to the complement of P in the space R of g -dimensional vector subspaces isotropic for the pencil.

Since the proof is on similar lines as in §4, [3], we only sketch it briefly with necessary modifications. We will actually show that the generalized Jacobian of Y is isomorphic to the complement of $P'(3.5)$ in the variety R' of $(g+2)$ -dimensional spaces on which Q has rank two generically. We take J as the variety of line bundles of degree g on Y .

$\bigoplus_{i=1}^{2g} C_i \oplus C_0$, length $\theta/I=2$, support $\theta/I=w_0$. Using $\alpha \otimes i^* \alpha = \theta_Y(g)$ and $\alpha_{w_i} = (i^* \alpha)_{w_i}$, one gets $H^0(N \otimes \theta_W) \approx \bigoplus C_i \oplus C_0$. Define $f(\alpha) = H^0(N)$ embedded in $\bigoplus C_i \oplus C_0$ as above. As in §4, [3] one can check that $f(\alpha)$ is in R' . If L is the kernel of the map $N \rightarrow N \otimes \theta/I$, $H^0(N) \cap \left(\bigoplus_{i=1}^{2g} C_i \right) = H^0(L)$ and the latter is of dimension g , so that $f(\alpha)$ is in $R' - P'$.

4.2. Definition of f^{-1} . Define a map $h: Y \times k^{2g+2} \rightarrow \theta_W$ by $h=0$ outside W , h/w_i is given by mapping C_i isomorphically onto θ/m_{w_i} and h/w_0 is given by mapping C_0 isomorphically onto θ/I . The kernel K of this map is a vector bundle (Lemma 2.2, [1]). Since Q/K vanishes identically on W , it induces a form q on K with values in $\theta_Y(2g+1)(-2W) = \theta_Y(-1)$ which is everywhere nondegenerate. Take V in $R' - P'$. Let V' be the fibre product of K and $Y \times V$ over $Y \times k^{2g+2}$. As the form restricted to $\bigoplus_{i>0} C_i$ is dual to a nondegenerate pencil, as in the case when Y is nonsingular, one

has $V + \left(\bigoplus_{i \neq j} C_j \right) + C_0 = k^{2g+2}$ for $i > 0$. By Remark 3.7, $V + \left(\bigoplus_{i>0} C_i \right) = k^{2g+2}$.

Therefore the composite of h with the inclusion of $Y \times V$ in $Y \times k^{2g+2}$ is onto, showing that V' is locally free. As in the nonsingular case [3], q restricted to V' has rank two everywhere. Let V'' be the kernel of q on V' . Define $F = (V'/V'')(g+1)$. The quadratic form on F gives an isomorphism $q: F \rightarrow F^*(2g+1)$. There is also an isomorphism $\psi: F \rightarrow F^*(2g+1)$ induced by the canonical alternating form on F with values in $\det F$. Define $N = \text{Ker}((q^{-1} \circ \psi) - \text{Id})$. Then $F = N \oplus i^* N$ with $i^* N \approx \text{Im}((q^{-1} \circ \psi) - \text{Id})$ (3.7, [3]). Let $L = (N \otimes \xi^{-1})(g)$. As in [3], to give the inverse of f , we have only to choose a canonical square root of the even degree line bundle L which can be done using Proposition 2.5.

Remark 4.3. The desingularization $\tilde{Y}$ of Y is a nonsingular hyperelliptic curve with ramification points $w_1, w_2, \dots, w_{2g}$. So, by the theorem in the nonsingular case, the Jacobian of Y is isomorphic to P_0 (defined in 3.5) and hence to P' .

5. The compactified Jacobian

We shall first prove the following theorem in the case of a single node and then indicate the general proof.

Theorem 2. Let Y be an irreducible curve of arithmetic genus g with only ordinary double points as singularities. Let $p: Y \rightarrow \mathbb{P}^1$ be a double cover such that the set of ramification points contains all double points. Then the compactified Jacobian $\bar{J}$ of Y is isomorphic to the variety R of g -dimensional subspaces of k^{2g+2} which are isotropic for the singular pencil of quadrics of the form

$$Q_1 = \sum_i X_i^2 + \sum_j 2Y_j Z_j$$

$$Q_2 = \sum_i a_i X_i^2 + \sum_j (Y_j^2 + 2b_j Y_j Z_j)$$

Proof. We first show that the morphism f^{-1} defined on $R' - P'$ (§4.2) can be extended to whole of R' . Starting with a V in R' we get a rank two coherent sheaf F as in 4.2. If V is in P' , F is not locally free, nor does it split. However, we do have a nondegenerate quadratic form on F with values in $\mathcal{O}_Y(2g+1) \otimes m_{w_0}$ in this case, giving an isomorphism $q: F \xrightarrow{\sim} F^*(2g+1) \otimes m_{w_0}$. Also, one has $\det F$ modulo torsion $= \mathcal{O}_Y(2g+1) \otimes m_{w_0}$. So the canonical alternating form on F gives a morphism (and not an isomorphism) $\Psi: F \rightarrow F^*(2g+1) \otimes m_{w_0}$. Define $N = \ker((q^{-1} \circ \psi) - Id)$. Notice that this definition of N agrees with that in the case V is in $R' - P'$, in that case we in fact have $F = \text{Ker}((q^{-1} \circ \psi) - Id) \oplus \text{Im}(q^{-1} \circ \psi - Id) = N \oplus i^*N$. The rest of the procedure to define f^{-1} works as before, i.e. one can choose canonically a torsion free rank 1 sheaf L such that $L \otimes L$ modulo torsion $= (N \otimes \xi^{-1})(g)$ and define $f'(V) = L$. Thus one gets a morphism $f': R' \rightarrow \bar{J}$ extending f^{-1} . One can check by similar methods that f' restricted to P' is an isomorphism onto $\text{Pic } \bar{Y}$ embedded in $\bar{J}$ as the set J_1 of torsion free rank one sheaves which are not locally free, the embedding being given by the map $L \rightarrow \pi_* L$. Thus we have a bijective birational morphism $f': R' \rightarrow \bar{J}$. Let $\tilde{R}$ and $\tilde{J}$ denote the normalizations of R' and $\bar{J}$ respectively. $\tilde{J}$ is the disjoint union of $\text{Pic } Y$ and two copies of $\text{Pic } \bar{Y}$, so $\tilde{J} \neq R'$. The morphism f' induces an isomorphism of normalizations $\tilde{R}$ and $\tilde{J}$, let $\tilde{f}$ be the composite of the inverse of this isomorphism with the natural map from $\tilde{R}$ to R' . Since $\tilde{f}$ respects the identification of two copies of $\text{Pic } \bar{Y}$, it factors through $\bar{J}$ giving $\tilde{f}: \tilde{J} \rightarrow R'$, the inverse of f' . This finishes the proof of the theorem in the one node case.

6. The general case

In the case of m nodes, we have a decomposition $k^{2g+2} = \bigoplus_{i=1}^n C_i \oplus \bigoplus_{j=1}^m D_j$, $\dim C_i = 1$, $\dim D_j = 2$, $n + 2m = 2g + 2$. It can be proved (in a similar way as in the one node case) that the generalized Jacobian of Y is isomorphic to the open subset of R' consisting of those subspaces which intersect $\bigoplus_i C_i \bigoplus_{j \neq k} D_j$, in subspaces of dimension g , for $i \leq k \leq m$. As before, we can also get a birational bijective morphism $\bar{f}: R' \rightarrow \bar{J}$ inducing an isomorphism of normalizations. We identify the normalizations $\tilde{R}$ and $\tilde{J}$ via this isomorphism. Thus we are again in the situation $\tilde{J} \rightarrow R' \rightarrow \bar{J}$. At a singular point L of $\bar{J}$, the complete local ring has the form $T \otimes \prod_i k[X_i, Y_i]/(X_i Y_i)$, i varying over the p number of nodes at which L is not locally free, T being a regular ring. Thus, the singularities of $\bar{J}$ being product singularities, we can prove $R' = \bar{J}$ using induction on p , i.e. can assume the result for $p < m$. Let U be the complement in $\bar{J}$ of the closed subset S corresponding to torsion free sheaves which are not locally free at all m nodes. By induction, $\mathcal{O}_{R'/U} \simeq \mathcal{O}_{\bar{J}/U}$. Since $\bar{J}$ is Cohen-Macaulay and $\bar{J} - U$ is not a divisor ($m \geq 2$) we have $H^0(V, \mathcal{O}_{\bar{J}}) \simeq H^0(U \cap V, \mathcal{O}_{\bar{J}})$, for any open set V . The commutative diagram

$$\begin{array}{ccc} H^0(V, \mathcal{O}_{\bar{J}}) & \xrightarrow{\sim} & H^0(U \cap V, \mathcal{O}_{\bar{J}}) \\ \downarrow & & \downarrow \approx \end{array}$$

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Decomposition of the de Rham complex

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Abstract. We give a reformulation of Deligne and Illusie's characteristic p proof of the degeneration of the “Hodge-to-de Rham” spectral sequence, which replaces patching in the derived category by an explicit quasi-isomorphism.

Keywords. de Rham complex; Hodge decomposition; liftings of varieties.

Deligne and Illusie [1] showed that if X is a smooth variety of dimension d over a perfect field k of characteristic p , such that $d < p$, and X has a lifting to a scheme flat (and hence smooth) over $W_2(k)$, the ring of Witt vectors of length 2 over k , then we have an isomorphism in the derived category of bounded complexes of coherent $\mathcal{O}_X$ modules

$$F_*\Omega_{X/k}^\bullet \cong \bigoplus_{i=0}^d \Omega_{X/k}^i[-i].$$

This easily implies the degeneration of the “Hodge-to-de Rham” spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_{X/k}^p) \Rightarrow H_{DR}^{p+q}(X/k),$$

as well as the vanishing theorem of Kodaira-Akizuki-Nakano

$$H^q(X, \Omega_{X/k}^p \otimes \mathcal{L}^{-1}) = 0 \quad \forall p + q < d$$

for an ample invertible sheaf $\mathcal{L}$ on X . This implies both statements for smooth proper varieties in characteristic 0 as well.

Our aim here is to give a quick proof of this result, using the following facts:

- (i) If X is smooth over k , and B_X^i, Z_X^i are the sheaves of locally exact and closed i -forms, respectively, with their natural $\mathcal{O}_X$ module structures (as $\mathcal{O}_X$ -submodules of $F_*\Omega_{X/k}^i$), and

$$0 \rightarrow B_X^i \rightarrow Z_X^i \xrightarrow{C} \Omega_{X/k}^i \rightarrow 0$$

(where C is the Cartier operator) has extension class

$$\xi_i \in \text{Ext}_X^1(\Omega_{X/k}^i, B_X^i)$$

is flat over $W_2(k)$, and $F^{(2)}: X^{(2)} \rightarrow X^{(2)}$ is a morphism, covering the Frobenius automorphism of $W_2(k)$, and restricting to the absolute Frobenius morphism of X . A proof of this is given in the appendix to [2].

- (ii) If we apply the functor $\text{Hom}_X(\Omega_{X/k}^1, -)$ to the exact sheaf sequence (d = exterior derivative)

$$0 \rightarrow \mathcal{O}_X \rightarrow F_* \mathcal{O}_X \xrightarrow{d} B_X^1 \rightarrow 0,$$

to obtain the exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}_X^1(\Omega_{X/k}^1, \mathcal{O}_X) \rightarrow \text{Ext}_X^1(\Omega_{X/k}^1, F_* \mathcal{O}_X) \xrightarrow{\alpha} \text{Ext}_X^1(\Omega_{X/k}^1, B_{X/k}^1) \\ \xrightarrow{\hat{\alpha}} \text{Ext}_X^2(\Omega_{X/k}^1, \mathcal{O}_X) \rightarrow \cdots \end{aligned}$$

then $\hat{\alpha}(\xi_1) \in \text{Ext}_X^2(\Omega_{X/k}^1, \mathcal{O}_X)$ is the obstruction to lifting X to a scheme $X^{(2)}$ flat over $W_2(k)$. This follows easily from the argument in the appendix to [2] (used in proving (i)).

Granting these statements, we see that if X is a smooth variety over k which has a flat lifting to a scheme over $W_2(k)$, then

$$\hat{\alpha}(\xi_1) = 0,$$

so that we have:

Lemma 1. If X lifts to a scheme flat over $W_2(k)$, then

$$\xi_1 = \alpha(\zeta)$$

for some $\zeta \in \text{Ext}_X^1(\Omega_{X/k}^1, F_* \mathcal{O}_X)$. Hence we have a pushout diagram of exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow & F_* \mathcal{O}_X & \rightarrow & \mathcal{V} & \rightarrow & \Omega_{X/k}^1 & \rightarrow 0 \\ & \downarrow d & & \downarrow & & \parallel & \\ 0 \rightarrow & B_X^1 & \rightarrow & Z_X^1 & \rightarrow & \Omega_{X/k}^1 & \rightarrow 0 \end{array}$$

for some locally free $\mathcal{O}_X$ -module $\mathcal{V}$.

The top row may be regarded as a quasi-isomorphism (which we denote ψ_1) between the 2-term complex (which we denote $\mathcal{C}_1$)

$$F_* \mathcal{O}_X \rightarrow \mathcal{V}$$

and

$$\Omega_{X/k}^1[-1].$$

The pushout diagram induces a map of complexes

$$\varphi_1: \mathcal{C}_1 \rightarrow F_* \Omega_{X/k}^1,$$

which induces an isomorphism on first cohomology sheaves. This is done as follows:

from β by composing with the inclusion $Z_X^1 \subset F_*\Omega_{X/k}^1$, then we have a diagram whose vertical maps give a map of complexes ($d = \dim X$)

$$\begin{array}{ccccccc} 0 \rightarrow F_*\mathcal{O}_X & \rightarrow \mathcal{V} & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow 0 \\ & \parallel & \downarrow \gamma & \downarrow & & \downarrow & \downarrow \\ 0 \rightarrow F_*\mathcal{O}_X & \rightarrow F_*\Omega_{X/k}^1 & \rightarrow F_*\Omega_{X/k}^2 & \rightarrow \cdots \rightarrow & F_*\Omega_{X/k}^d & \rightarrow 0. \end{array}$$

The complex $\mathcal{C}_i^{\otimes i}$ is clearly quasi-isomorphic to the complex $(\Omega_{X/k}^1)^{\otimes i}[-i]$; the multiplicative structure on the de Rham complex yields a map " $\varphi_i^{\otimes i}$ " of complexes

$$C_1^{\otimes i} \rightarrow F_*\Omega_{X/k}^i,$$

which yields the exterior product map

$$\Lambda^i: (\Omega_{X/k}^1)^{\otimes i} \rightarrow \Omega_{X/k}^i$$

on i -th cohomology sheaves. If $i < p$, the characteristic is zero, then as in [1], we see that the map Λ^i is a *split surjection*; hence we can find an injective $\mathcal{O}_X$ -linear map

$$h_i: \Omega_{X/k}^i \rightarrow (\Omega_{X/k}^1)^{\otimes i}$$

which splits the surjection Λ^i . Let $\mathcal{C}_i$ be the complex of locally free $\mathcal{O}_X$ -modules obtained from the pullback diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow (\mathcal{C}_i)^0 & \cdots & (\mathcal{C}_i)^{i-1} & \rightarrow & (\mathcal{C}_i)^i & \rightarrow & \Omega_{X/k}^i \rightarrow 0 \\ & \parallel & & & \downarrow & & h_i \downarrow \\ 0 \rightarrow (\mathcal{C}_1^{\otimes i})^0 & \cdots & (\mathcal{C}_1^{\otimes i})^{i-1} & \rightarrow & (\mathcal{C}_1^{\otimes i})^i & \rightarrow & (\Omega_{X/k}^1)^{\otimes i} \rightarrow 0 \end{array}$$

Thus, there is a quasi-isomorphism

$$\psi_i: \mathcal{C}_i \rightarrow \Omega_{X/k}^i[-i],$$

and " $\varphi_1^{\otimes i}$ " induces a map of complexes

$$\varphi_i: \mathcal{C}_i \rightarrow F_*\Omega_{X/k}^i$$

which yields an isomorphism on the i th cohomology sheaves.

Hence, if $d < p$, the map

$$\sum_{i=0}^d \varphi_i: \bigoplus_{i=0}^d \mathcal{C}_i \rightarrow F_*\Omega_{X/k}^i$$

is a quasi-isomorphism (define $\mathcal{C}_0$ to be $\mathcal{O}_X$ in degree 0, and φ_0 to be the map induced by the inclusion $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$). Since there is a quasi-isomorphism

$$\bigoplus_{i=0}^d \psi_i: \bigoplus_{i=0}^d \mathcal{C}_i \rightarrow \bigoplus_{i=0}^d \Omega_{X/k}^i[-i],$$

we deduce that there is an isomorphism in the derived category

$$F_*\Omega_{X/k}^i \rightarrow \bigoplus_{i=0}^d \Omega_{X/k}^i[-i].$$

Remark. The argument may be made more elementary by giving a direct proof of

the lemma, which does not appeal to the results in the Appendix of [2] (however, the proof given there is along the same lines as the argument given below; it is also the main point of the argument in [1], apart from the patching argument).

If A is a smooth k -algebra of finite type, and B is a flat $W_2(k)$ -algebra lifting A i.e. satisfying $B \otimes_{W_2(k)} k \cong A$, then for any $W_2(k)$ -algebra C , any k -algebra homomorphism $f: A \rightarrow C \otimes_{W_2(k)} k$ "lifts" to a $W_2(k)$ -algebra homomorphism $B \rightarrow C$. Hence, if X is a smooth k -variety which lifts to a flat $W_2(k)$ -scheme Y , then for each affine open set $U \subset Y$, the absolute Frobenius morphism of $V = X \cap U$ lifts to a $W_2(k)$ -semi-linear morphism $U \rightarrow U$ (take $B = H^0(\mathcal{O}_U)$, $A = H^0(\mathcal{O}_V)$, and $C = H^0(F_* \mathcal{O}_U)$).

If $U = \text{Spec } B$, $V = \text{Spec } A$, then such a lifting of Frobenius corresponds to the choice of a function $\varphi: B \rightarrow A$ such that, identifying A with pB , ($p = \text{characteristic of } k$), we have formulae

$$\varphi(a+b) = \varphi(a) + \varphi(b) - \sum_{j=1}^{p-1} \left(\binom{p}{j} / p \right) \bar{a}^j \bar{b}^{p-j},$$

$$\varphi(ab) = \bar{a}^p \varphi(b) + \bar{b}^p \varphi(a),$$

for all $a, b \in B$, where $\bar{a}, \bar{b}$ denote the images of a, b in A . The lifting of Frobenius is given by the map $B \rightarrow B$, $a \mapsto a^p + p\varphi(a)$. (Note that

$$\binom{p}{j} / p$$

is an integer.)

Given such a lifting of Frobenius, we have a derivation $A \rightarrow Z_A^1$ (where Z_A^1 is the A -module corresponding to $Z_{\text{Spec } A}^1$), given by $a \mapsto a^{p-1} da + d\varphi(\bar{a})$, where $\bar{a} \in B$ lifts a . This is well defined, and induces an A -linear map $\Omega_{A/k}^1 \rightarrow Z_A^1$ which splits the surjection given by the Cartier operator.

Hence if $\{U_i = \text{Spec } B_i\}$ is an affine open cover of $X^{(2)}$, $\{V_i = \text{Spec } A_i\}$ the corresponding open cover of X , and $\varphi_i: B_i \rightarrow B_i$ are lifts of Frobenius, then on $V_i \cap V_j = \text{Spec } A_{ij}$, the derivation $a \mapsto d\varphi_i(\bar{a}) - d\varphi_j(\bar{a})$ yields an A_{ij} -linear map $c_{ij}: \Omega_{A_{ij}/k}^1 \rightarrow B_{A_{ij}}^1$, such that $\{c_{ij}\}$ gives a Čech 1-cocycle representing the extension class

$$\xi_1 \in \text{Ext}_X^1(\Omega_{X/k}^1, B_X^1).$$

But then $a \mapsto \varphi_i(\bar{a}) - \varphi_j(\bar{a})$ is a derivation $A_{ij} \rightarrow F_* A_{ij}$, giving a map $b_{ij}: \Omega_{A_{ij}}^1 \rightarrow F_* A_{ij}$, and $\{b_{ij}\}$ is a 1-cocycle representing a class

$$\zeta \in \text{Ext}_X^1(\Omega_{X/k}^1, F_* \mathcal{O}_X)$$

satisfying $\alpha(\zeta) = \xi_1$. The existence of such a class ζ is precisely the assertion of the lemma.

On the ratio of two blocks of consecutive integers

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Abstract. Under certain assumptions, it is shown that eq. (2) has only finitely many solutions in integers $x \geq 0$, $y \geq 0$, $k \geq 2$, $l \geq 0$. In particular, it is proved that (2) with $a=b=1$, $l=k$ implies that $x=7$, $y=0$, $k=3$.

Keywords. Exponential Diophantine equations; elementary Diophantine approximations; linear forms in logarithms; arithmetic-geometric mean.

1.

Let a and b be relatively prime positive integers. Erdős [5] conjectured that there are only finitely many integers $x \geq 0$, $y \geq 0$, $k \geq 2$, $l \geq 0$ with $k+l \geq 3$ satisfying

$$x \geq y + l + k \quad (1)$$

and

$$a(x+1) \cdots (x+k) = b(y+1) \cdots (y+k+l). \quad (2)$$

By letters x, y, k and l , we shall always understand that $x \geq 0$, $y \geq 0$, $k \geq 2$, $l \geq 0$ with $k+l \geq 3$ are integers satisfying (2). For an integer v with $|v| > 1$, we denote by $P(v)$ and $\omega(v)$, respectively, the greatest prime factor of v and the number of distinct prime factors of v . Further, we write $P(0)=P(1)=P(-1)=1$ and $\omega(0)=\omega(1)=\omega(-1)=0$.

In this and the next paragraph, we state some of the earlier results on (2). Mordell [7] proved that (2) with $a=b=1$, $k=2$, $l=1$ implies that either $x=1$, $y=0$ or $x=13$, $y=4$. Avanesov [1] confirmed a conjecture of Sierpinski by proving that $x=0$, $y=0$; $x=3$, $y=2$; $x=14$, $y=7$; $x=54$, $y=19$ and $x=118$, $y=33$ are the only solutions of (2) with $a=3$, $b=1$, $k=2$, $l=1$. Tzanakis and de Weger [12] determined all the solutions of (2) with $a=1$, $b=2$, $k=2$, $l=1$. Boyd and Kisilevsky [3] showed that $x=1$, $y=0$; $x=3$, $y=1$ and $x=54$, $y=18$ are the only solutions of (2) with $a=b=1$, $k=3$, $l=1$. Cohn [4] proved that (2) with $a=1$, $b=2$, $k=4$, $l=0$ is satisfied only if $x=4$, $y=3$. Further, Ponnudurai [8] showed that $x=2$, $y=1$ and $x=6$, $y=4$ are the only solutions of (2) with $a=1$, $b=3$, $k=4$, $l=0$. Shorey [10] showed that (2) with $l=0$ and (1) implies that either k is bounded by an effectively computable number depending only on a , b or $k = [\alpha + 1]$ where

$$\alpha = \log \left(\frac{b}{a} \right) / \log \left(\frac{x}{y} \right).$$

an effectively computable number depending only on $a, b, P(x), P(y)$ and also,

$$\log x \leq C_1 k \quad (3)$$

where C_1 is an effectively computable number depending only on $a, b, P(y)$ and $P(x-y)$. On the other hand, we see from Cramer's conjecture on distance between consecutive primes that (2) and (1) imply that

$$(\log x)^2 > C_2 k \quad (4)$$

where $C_2 > 0$ is an absolute constant. In this paper, we shall extend these results to a more general equation (2). For a given k and l , we refer to a theorem of Siegel [11] to observe that (2) has only finitely many solutions in x and y provided that the curve represented by (2) is irreducible over the field of complex numbers and has positive genus.

It has not been possible to confirm the conjecture of Erdős, stated above, even when y is bounded. Erdős [5] considered a particular case of (2) corresponding to $y=0$, namely,

$$(X+1) \cdots (X+M) = N! \quad (5)$$

where $X \geq 2$, $M \geq 2$ and $N \geq 2$ are integers. Erdős[5] conjectured that $8.9.10=6!$ is the only solution of (5). Erdős[5] proved that for $\varepsilon > 0$ there exists N_0 depending only on ε such that (5) with $N \geq N_0$ implies that

$$X \geq (2-\varepsilon)^N. \quad (6)$$

We re-write (5) with $M=2$ as

$$(2X+3)^2 - 1 = 4N!$$

which reminds us of the open problem on squares of the form $N! + 1$.

By fixing any three of the four variables x, y, k and l in (2), the fourth one is determined uniquely, if it exists. For given x, y and l , we start with the following result that determines the exact value of k if it exists.

Theorem 1. Let $y' = \max(y, 1)$ and

$$\beta = \left(\log \left(\frac{b}{a} \right) + l \log y' \right) / \log \left(\frac{x}{y'} \right). \quad (7)$$

There exists an effectively computable number C_3 depending only on a and b such that (2) with (1), $k \geq C_3$ and

$$y > (k+l)^2 \text{ if } 12 \log(l+1) \geq k \quad (8)$$

implies that

$$0 < k - \beta < 1. \quad (9)$$

We observe that a restriction of the type (8) is necessary for obtaining (9). For

$0 < C_4 < 1$ is a number depending only on a, b and ϕ .

Next, we turn to a more general situation than considered in Theorem 1. For given x and y , we denote by $N(x, y)$ the number of pairs (k, l) satisfying (2) and (1). For given x, y and l , there is at most one k satisfying (2). Therefore, we see from Lemma 1 that

$$N(x, y) \leq C_5 \log x \quad (10)$$

where C_5 is an effectively computable number depending only on a and b . In the following result, we sharpen (10) whenever y is somewhat smaller than x .

Theorem 2.(a) *There exist effectively computable numbers C_6 and C_7 depending only on a and b such that for every x and y with*

$$\log y \leq C_6 \frac{\log x}{\log \log x}, \quad (11)$$

we have

$$N(x, y) \leq C_7. \quad (12)$$

(b) *Let $\varepsilon > 0$ and $y < (1 - \varepsilon)x$. Then*

$$N(x, y) \leq C_8 \log \log x$$

where C_8 is an effectively computable number depending only on a, b and ε .

A pair (k, l) in Theorem 2 may depend on x and y . Now, for $\varepsilon > 0$ and $y \leq x^{1-\varepsilon}$, we show that $\max(k, l)$ is bounded by a number depending only on ε, a, b and $P(x)$. See Theorem 3(b) which finds an application in the proof of Theorem 5. Furthermore, we give lower bounds for $P(x)$ and $P(x-y)$ whenever y is smaller than some power of x and we apply these estimates in the proof of Corollary 1.

Theorem 3. *Suppose that (2) with (1) is satisfied. Let $\varepsilon > 0$.*

(a) *If*

$$x - y \geq \min \left(\varepsilon x, \frac{x((\log k)(\log \log k))^2}{k} \right), \quad (13)$$

then l is bounded by an effectively computable number depending only on ε, a, b and $P(x)$.

(b) *If $y \leq x^{1-\varepsilon}$, then $\max(k, l)$ is bounded by an effectively computable number depending only on ε, a, b and $P(x)$.*

(c) *There exist effectively computable numbers $\delta > 0$ and C_9 depending only on a and b such that the inequalities*

$$y \leq x^\delta \quad \text{and} \quad P(x) \leq l + k$$

imply that $\max(x, y, k, l) \leq C_9$.

(d) *There exist effectively computable numbers $\delta_1 > 0$ and C_{10} depending only on a, b and $P(x-y)$ such that if $y \leq x^{\delta_1}$, then $\max(x, y, k, l) \leq C_{10}$.*

(e) *There is C_{11} depending only on a, b and y such that $P(x-y) \leq l$ implies that $\max(x, k, l) \leq C_{11}$.*

The proof of Theorem 3(a) depends on the theory of linear forms in logarithms. We combine Theorem 3(c), (d) with Lemma 8. We derive that (2) with (1) and $P(x) \leq l+k$ implies that l/k is bounded by an effectively computable number depending only on a and b . Also, we see that (2) and (1) imply that l/k is bounded by an effectively computable number depending only on a , b and $P(x-y)$. Furthermore, if k is fixed, we may apply the theorem of Siegel, stated above, to finitely many pairs (k, l) given by the above two assertions to derive that (2) has only finitely many solutions in x and y under certain assumptions already mentioned. Next, we consider (2) with $l/k = 1$ and $a=b=1$ i.e.

$$(x+1)\cdots(x+k)=(y+1)\cdots(y+2k). \quad (14)$$

MacLeod and Barrodale [6] showed that (14) with $k \in \{2, 4, 5\}$ has no solution in x, y and (14) with $k=3$ implies that $x=7, y=0$. Further, MacLeod and Barrodale [6] proved that for a given k , there are only finitely many pairs (depending on k) x, y satisfying (14). We prove

Theorem 4. *The equation (14) has only one solution in integers $x \geq 0, y \geq 0, k \geq 2$ and it is given by $x=7, y=0, k=3$.*

In Theorems 2, 3 and 4, we see that y is somewhat smaller than x . In the next result, we replace this by the assumption that $P(y)$ is bounded. We combine Theorem 3(b) with the theory of linear forms in logarithms to obtain the following result.

Theorem 5. *Equation (2) with (1) implies that $\max(x, y, k, l)$ is bounded by an effectively computable number depending only on $a, b, P(x)$ and $P(y)$.*

We may derive from Theorem 3(c) that (5) implies $P(X) > N$ whenever X exceeds a sufficiently large effectively computable absolute constant. In fact, we apply Theorem 3(e) to obtain a more general result on (5).

COROLLARY 1.

Let $B \geq 0$ be an integer. There exists an effectively computable number C_{12} depending only on B such that (5) with $N > C_{12}$ implies that $P(X-B) > N$.

For the proofs of our Theorems, we prove certain estimates that are of independent interest. For example, we show unconditionally that (4) is valid whenever $y < (1-\varepsilon)x$ for $\varepsilon > 0$. We formulate these estimates as the following theorem.

Theorem 6. *Suppose that (2) with (1) is satisfied. Then*

(a) *There exists an effectively computable number $C_{13} > 0$ depending only on a and b such that*

$$x \geq C_{13} k^3 (\log k)^{-4}. \quad (15)$$

(b) *Let $\varepsilon > 0$ and $y < (1-\varepsilon)x$. Then*

$$(\log x)^2 > C_{14} k \quad (16)$$

where $C_{14} > 0$ is an effectively computable number depending only on ε, a and b .

(c) There exist effectively computable numbers C_{15} and $C_{16} > 0$ depending only on a and b such that if $k \geq C_{15}$ and $y \leq (k+l)^3 (\log(k+l))^{-5}$, we have

$$\log \log x > C_{16}k.$$

(d) There exists an effectively computable number $C_{17} > 0$ depending only on a and b such that

$$x - y \geq C_{17}x^{2/3}.$$

2.

This section contains a proof of Theorem 6 and lemmas for our Theorems 1, 2, 3, 5. Throughout this section, we suppose that (1) and (2) are satisfied and we shall use this assumption without reference. We put

$$U_i = ax^i - by^{i+1} \quad \text{for } 0 \leq i \leq k \quad (17)$$

and

$$f(x, y) = x - (b/a)^{1/k} y^{1+(l/k)}. \quad (18)$$

Let

$$F(z) = (z+1) \cdots (z+k) = z^k + A_1 z^{k-1} + \cdots + A_k \quad (19)$$

and

$$G(z) = (z+1) \cdots (z+l) = z^l + A'_1 z^{l-1} + \cdots + A'_l. \quad (20)$$

Then, we refer to [6, p. 256] to observe that

$$0 < A_i \leq (k+1)^{2i/2^i i!} \quad \text{for } 1 \leq i \leq k \quad (21)$$

and

$$0 < A'_j \leq (l+1)^{2j/2^j j!} \quad \text{for } 1 \leq j \leq l. \quad (22)$$

We start with the following result that provides an upper bound for l .

Lemma 1. There exists an effectively computable absolute constant c such that

$$l \leq c \log(a+1) + (2 \log x) / \log 2. \quad (23)$$

Proof. We re-write (2) as

$$a \frac{(x+1) \cdots (x+k)}{k!} = b \frac{(y+1) \cdots (y+k+l)(k+l)!}{(k+l)! k!}. \quad (24)$$

We observe that

$$\text{ord}_2(\text{R.H.S. of (24)}) \geq (l-1)/2. \quad (25)$$

On the other hand,

$$\text{ord}_2(\text{L.H.S. of (24)}) \leq \text{ord}_2(a) + \max_{1 \leq i \leq k} \text{ord}_2(x+i) \leq (\log(a(x+k))) / \log 2.$$

We give estimates for $f(x, y)$ in the next three lemmas.

Lemma 2.

$$f(x, y) > 0.$$

Proof. We suppose that $f(x, y) \leq 0$. Then, we see from (17) and (18) that $U_k \leq 0$, which since $x > y$, implies that

$$U_i \leq 0 \quad \text{for } 0 \leq i \leq k.$$

Now, we derive from (2), (19), (20), (17) and (28) that

$$0 < aF(x) - bF(y)y^l = U_k + A_1 U_{k-1} + \cdots + A_k U_0 \leq 0$$

which is a contradiction.

By (2), we observe that $ax^k < b(y+k+l)^{k+l}$ which implies that

$$x < (b/a)^{1/k} (y+k+l)^{1+(l/k)} \leq \theta (y+k+l)^{1+(l/k)}$$

where

$$\theta = \max(1, (b/a)^{1/k}) \leq \max(1, (b/a)).$$

Now, we give an upper estimate for $f(x, y)$.

Lemma 3. For $l \leq k$, we have

$$f(x, y) \leq 16\theta k (\max(k, y))^{l/k}.$$

Proof. Suppose that $y > k+l$. Then, by (30) and $l \leq k$, we have

$$\begin{aligned} x &< (b/a)^{1/k} y^{1+(l/k)} \left(1 + \frac{k+l}{y}\right)^{1+(l/k)} \\ &\leq (b/a)^{1/k} y^{1+(l/k)} \left(1 + \frac{(k+l)^2}{ky} + \frac{l(k+l)^3}{2k^2 y^2}\right) \end{aligned}$$

which implies that

$$f(x, y) \leq 6\theta k y^{l/k}.$$

If $y = k+l$, we see from (30) and $l \leq k$ that

$$f(x, y) \leq 3\theta y^{1+(l/k)} \leq 6\theta k y^{l/k}.$$

Thus, we may suppose that $y < k+l$. Then

$$\begin{aligned} x &< \theta (k+l)^{1+(l/k)} \left(1 + \frac{y}{k+l}\right)^{1+(l/k)} \\ &\leq \theta (k+l)^{1+(l/k)} \left(1 + \frac{y}{k} + \frac{ly^2}{2k^2(l+k)}\right) \end{aligned}$$

which implies that

$$f(x, y) < x \leq 16\theta k^{1+(l/k)}. \quad \square$$

For applications, it is convenient to formulate a version of Lemma 3 which is valid also for $l > k$.

Lemma 4.

$$f(x, y) \leq 16\theta k x^{l/k}. \quad (31)$$

Proof. By (1), we observe that $x > \max(k, y)$. Now, we apply Lemma 3 to assume that $l > k$. Then, the trivial estimate $f(x, y) < x$ implies (31). $\square$

We apply our estimates on $f(x, y)$ to give bounds for $x - y$.

Lemma 5. There exist effectively computable numbers c_1 and $c_2 > 0$ depending only on a and b such that

$$x - y \leq \begin{cases} c_1 \left(k + \frac{lx \log x}{k} \right) & \text{if } l > 0 \\ c_1 \left(k + \frac{x}{k} \right) & \text{if } l = 0 \end{cases} \quad (32)$$

and

$$x - y \geq \begin{cases} (c_2 ly \log(y+1))/k, & \text{if } l > 0 \\ c_2 y/k & \text{if } l = 0. \end{cases} \quad (33)$$

Proof. We write c_3, c_4 and c_5 for effectively computable numbers depending only on a and b . We write, by (18),

$$x - y = f(x, y) + \Delta \quad (34)$$

where

$$\Delta = (b/a)^{1/k} y^{1+(l/k)} - y. \quad (35)$$

First, we prove (32). If $l = 0$, the assertion follows from (34), (35) with $b > a$ and (31). Let $l > 0$. Then, we may assume that

$$l \log x < k, \quad (36)$$

otherwise (32) follows immediately. Now, we derive from (31) and (36) that

$$f(x, y) \leq 16\theta ek. \quad (37)$$

Further, it is easy to see from (35) and (36) that

$$\Delta \leq (c_3 lx \log x)/k. \quad (38)$$

Finally, we combine (34), (37) and (38) to obtain (32).

Next, we turn to the proof of (33). By (34) and (27), we observe that

$$x - y \geq \Delta. \quad (39)$$

we may suppose that $y > 0$ and $l \log(y+1) > c_4$ with c_4 sufficiently large. Then, it is easy to see that $\Delta \geq (c_5 l y \log y)/k$ which, together with (39), implies (33).

We apply an argument of Erodös and Lemmas 1, 5 to obtain a lower bound for x in the next lemma. The case $l=0$ of this lemma is proved in [10].

Lemma 6. *There exists an effectively computable number $c_6 > 0$ depending only on a and b such that*

$$x(\log x)^2 \geq \begin{cases} c_6 k^3 l^{-2} & \text{if } l > 0 \\ c_6 k^3 (\log k)^2 & \text{if } l = 0. \end{cases} \quad (46)$$

Proof. We denote by $c_7, \dots, c_{13}$ effectively computable positive numbers depending only on a and b . We may assume that $k \geq c_7$ with c_7 sufficiently large and

$$x < k^3 \quad (41)$$

which, together with (23), implies that

$$l \leq c_3 \log k. \quad (42)$$

Furthermore, we derive from (2) and (1) that none of $x+1, \dots, x+k$ is a prime number. Therefore, it follows from the well-known results on difference between consecutive primes that

$$x \geq k^{3/2}. \quad (43)$$

We denote by d the greatest common divisor of $(x+1) \cdots (x+k)$ and $(y+1) \cdots (y+k+l)$. Then, by (2), we see that

$$x^k < (x+1) \cdots (x+k) \leq bd. \quad (44)$$

Let $S = \{x+1, \dots, x+k\}$. For a prime $p \leq k$, we choose an $f(p) \in S$ such that p does not appear to a higher power in the factorization of any other element of S . Let S_1 be the subset of S obtained by deleting all $f(p)$ with $p \leq k$. Then, by a fundamental argument of Erdös, we have

$$\prod_{s \in S_1} s \leq \prod_{p \leq k} p^{[(k/p)] + [(k/p^2)] + \dots} = k!.$$

Therefore, the contribution d_1 in d from all primes not exceeding k is at most

$$k^k (x+k)^{\pi(k)} \leq (e^4 k)^k,$$

by (41). Further, the contribution d_2 in d from all primes p with $k < p \leq (2k+l-1)$ is less than or equal to

$$(x+k)^{\pi(2k+l-1) - \pi(k)} \leq e^{4k},$$

by (41) and (42). Now, we set

$$d_3 = d/d_1 d_2 \text{ and } \Delta_1 = \left(\prod_{\mu = -(k+l-1)}^{k-1} (x-y+\mu) \right) / (2k+l-1)!.$$

Notice that Δ_1 is a positive integer not exceeding

$$e^{3k} \left(\frac{x-y}{2k} + 1 \right)^{2k+l-1},$$

by (42). Also, we observe from (2) that $d_3 | \Delta_1$. Consequently,

$$d \leq (c_9 k)^k \left(\frac{x-y}{2k} + 1 \right)^{2k+l-1}$$

which, together with (44) and (42), implies that

$$x \leq c_{10} k \left(\frac{x-y}{2k} + 1 \right)^{2+c_8(\log k)/k} \quad (45)$$

If $l > 0$, we combine (45), (43) and (32) to conclude that

$$x \leq c_{11} k \left(\frac{lx \log x}{k^2} \right)^{2+c_8(\log k)/k} \leq c_{12} k \left(\frac{lx \log x}{k^2} \right)^2,$$

by (41) and (42). If $l = 0$, we obtain in a similar way that $x \leq c_{13} k (x/k^2)^2$. The preceding two inequalities imply (40) immediately. $\square$

Now, we are ready to prove Theorem 6(a), (b), (d).

Proof of Theorem 6(a), (b), (d). First, we observe that (15) is an immediate consequence of (40) and (23). We re-write $y < (1-\varepsilon)x$ as $x-y > \varepsilon x$ which, together with (32), (15) and (23), implies (16). Finally, we apply (33) and (40) to obtain Theorem 6(d). $\square$

For applications, it is convenient to combine (32) and (15) to formulate the following result.

COROLLARY 2.

There exists an effectively computable number c_{14} depending only on a and b such that

$$x-y \leq (c_{14}(l+1)x \log x)/k. \quad (46)$$

For the proof of Theorem 6(c), we apply (15), (16) and (23) to obtain the following result which also finds an application in the proof of Theorem 1.

Lemma 7. Let $\chi \geq 1$ and $\gamma > 4/\log 2$. There exist effectively computable numbers c_{15} and c_{16} depending only on a , b , χ and γ such that for $k \geq c_{15}$ and

$$\log(l+1) < (\gamma\chi)^{-1}k, \quad (47)$$

we have

$$y \geq \begin{cases} c_{16} l^\chi & \text{if } l > k \\ c_{16} k^3 (\log k)^{-4} & \text{if } l \leq k. \end{cases}$$

Proof. We may assume that c_{15} is sufficiently large. Let $l > k$. Then, we may suppose

where c_{17} and the subsequent letter $c_{18} > 0$ are effectively computable numbers depending only on a and b . Therefore, by (23), we derive that $k \leq (\gamma\chi) \log l$ which contradicts (47). Thus, we may suppose that $l \leq k$. Now, we may assume that $y \leq k^3 (\log k)^{-4}$. Then, we observe from (30) that $k \geq x^{1/6}$ which, by Theorem 6(b) with $\varepsilon = 1/2$ and (15), implies that

$$y \geq x/2 \geq c_{18} k^3 / (\log k)^4. \quad \square$$

Proof of Theorem 6(c). We may suppose that C_{15} is sufficiently large. Then, we apply Lemma 7 with $\chi = 3$, $\gamma = 6$ and (23) to obtain

$$c_{19} \log \log x \geq \log(l+1) \geq k/18. \quad \square$$

Next, we prove a lemma which tells that certain assumptions involving variables of (2) are equivalent.

Lemma 8(a). Let $0 < \delta < 1$. There exist effectively computable numbers v_1 and $v_2 > 0$ depending only on δ such that $l \geq v_2 k$ whenever $x \geq v_1$ and $y \leq x^\delta$.

(b) Let $\mu > 0$. There exist effectively computable numbers v_3 and v_4 with $0 < v_4 < 1$ depending only on μ such that $y \leq x^{v_4}$ whenever $x \geq v_3$ and $l \geq \mu k$.

Proof. (a). We may assume that v_1 is sufficiently large. Let $y \leq x^\delta < x/2$. Then, we apply Theorem 6(b) with $\varepsilon = 1/2$ to obtain (16) which, together with (30) and (23), implies that $l \geq v_2 k$.

(b) By (2), $y < (a/b)^{1/(k+l)} (x+k)^{k/(k+l)}$ which implies the assertion. $\square$

Finally, we state an estimate of Baker [2] on linear forms in logarithms and its p -adic analogue, due to Yu [13]. Let $\alpha_1, \dots, \alpha_n$ be non-zero rational numbers of heights* not exceeding $A_1, \dots, A_n$, respectively. We assume that $A_j \geq 3$ for $1 \leq j \leq n$. We put

$$\Omega = \prod_{j=1}^n \log A_j, \quad \Omega' = \Omega / \log A_n.$$

Then we have

Lemma 9. (Baker [2]). There exists an effectively computable number c_{20} depending only on n such that the inequalities

$$0 < |\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| < \exp(-c_{20} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \dots, b_n$ of absolute values not exceeding $B (\geq 2)$.

Lemma 10. (Yu [13]). Let p be a prime number. Suppose that $b_1, \dots, b_{n-1}$ and $b_n = -1$ are rational integers of absolute values not exceeding $B (\geq 2)$. There exists an effectively computable number c_{21} depending only on n and p such that either $\alpha_1^{b_1} \cdots \alpha_n^{b_n} = 1$ or

$$\text{ord}_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) \leq c_{21} \Omega \log \Omega' \log B.$$

*The height of a rational number u_1/u_2 with $\gcd(u_1, u_2) = 1$ is defined as $\max(|u_1|, |u_2|)$.

Proof of Theorem 1. We denote by $c_{22}, c_{23}, \dots, c_{29}$ effectively computable positive numbers depending only on a and b . We may assume that $k \geq c_{22}$ with c_{22} sufficiently large. Let $F(z)$, $G(z)$ and U_i with $0 \leq i \leq k$ be given by (19), (20) and (17), respectively. We apply Lemma 7 with $\chi > 2$, $4/\log 2 < \gamma < 6$ and $\gamma\chi = 12$ to derive from (8) that

$$y > (k + l)^2. \quad (48)$$

By (27), we observe that $U_k > 0$. Therefore, it suffices to show that $U_{k-1} < 0$. We assume that

$$U_{k-1} \geq 0 \quad (49)$$

and we shall arrive at a contradiction. By (2),

$$aF(x) - bF(y)y^l = bF(y)R(y) \quad (50)$$

where $R(y) = 0$ if $l = 0$ and for $l > 0$,

$$R(y) = G(y + k) - y^l = R_1(y) + R_2(y)$$

with

$$R_1(y) = (y + k)^l - y^l, \quad R_2(y) = A'_1(y + k)^{l-1} + \dots + A'_l.$$

Then, it is easy to derive from the estimates (21), (22) and (48) that

$$F(y) \leq 3y^k, \quad R_1(y) \leq 2ky^{l-1}, \quad R_2(y) \leq 2l^2y^{l-1}.$$

Therefore, we derive from (50) that

$$aF(x) - bF(y)y^l = bF(y)R(y) \leq 6bl(k + l)y^{k+l-1}. \quad (51)$$

We notice from (17) that

$$U_k - xU_{k-1} = by^{k+l-1}(x - y) > 0$$

which, by (49), (29) and (50), implies that

$$by^{k+l-1}(x - y) \leq U_k = bF(y)R(y) - A_1U_{k-1} - \dots - A_kU_0. \quad (52)$$

Now, by (49), (21) and (48), we derive that

$$-A_1U_{k-1} - \dots - A_kU_0 \leq -A_2U_{k-2} - \dots - A_kU_0 \leq c_{23}k^4y^{k+l-2}$$

which, together with (51) and (52), implies that

$$x - y \leq c_{24}(kl + l^2 + k^4y^{-1}). \quad (53)$$

If $l = 0$, we see from (33) and (53) that $y \leq k^{1/4}$ which contradicts Lemma 7. Thus, we may assume that $l > 0$. Now, we combine again (33) and (53) to derive that

$$y \log y \leq c_{25}(k^2 + kl + k^5(l y)^{-1}). \quad (54)$$

Now, we combine (54) and (48) to observe that $l^2 \leq c_{25}(k^2 + kl + k^3 l^{-1})$ which implies that

$$l \leq c_{26}k. \quad (55)$$

Now, we see from (54), (48) and (55) that

$$y^2 \log y \leq c_{27}k^5 l^{-1}. \quad (56)$$

Then, we apply Theorem 6 to derive that

$$(\log x)^2 \geq c_{28}k. \quad (57)$$

Finally, we combine (1), (30), (57), (56) and (55) to conclude that $k < x \leq c_{29}$. $\square$

Proof of Theorem 2 (a). We write $c_{30}, c_{31}, \dots, c_{37}$ for effectively computable positive numbers depending only on a and b . Suppose that the assertion (12) is not valid. For a given k , we observe that there is at most one l satisfying (2). Therefore, we may assume that (1) and (2) are satisfied with $k = k_1$, $l = l_1$ and $k = k_2$, $l = l_2$ such that $k_1 < k_2$, $k_1 \geq c_{30}$ and $k_2 - k_1 \geq c_{30}$ with c_{30} sufficiently large. Then, by (2) with $k = k_1$, $l = l_1$ and $k = k_2$, $l = l_2$, we observe that

$$(X+1) \cdots (X+k_2-k_1) = (Y+1) \cdots (Y+k_2-k_1+l_2-l_1) \quad (58)$$

where

$$X = x + k_1, \quad Y = y + k_1 + l_1. \quad (59)$$

By (59) and (1), we see that $X > Y$ which, together with (58), implies that $l_1 < l_2$. Further, by (30), (11), (23) and Theorem 6 (b), we derive that

$$k_i \log \log x \leq c_{31}(l_i + 1) \quad (i = 1, 2) \quad (60)$$

which, together with (23), implies that

$$k_i \leq c_{32}(\log x) / \log \log x \quad (i = 1, 2). \quad (61)$$

Now, we see from (58), (59), (11), (61) and (23) that

$$(k_2 - k_1) \log \log x \leq c_{33}(l_2 - l_1). \quad (62)$$

By counting the power of 2 on both the sides of (24) with $a = b = 1$, $x = X$, $y = Y$, $k = k_2 - k_1$ and $l = l_2 - l_1$ obtained from (58), we have

$$l_2 - l_1 \leq 2 \max_{1 \leq i \leq k_2 - k_1} \text{ord}_2(X + i) + 1. \quad (63)$$

We show that

$$\text{either } l_1 \leq c_{34} \log k_2 \quad \text{or} \quad l_2 - l_1 \leq c_{34} \log k_2. \quad (64)$$

If

$$\max \text{ord}_2(x + i) = \text{ord}_2(x + i_0) \leq 2 \log k, \quad (65)$$

$l_1 \leq c_{35} \log k_2$. Therefore, we may suppose that (65) is not valid. Then, by (59), we write

$$X + i = x + j_0 + k_1 + i - j_0$$

to observe that

$$\text{ord}_2(X + i) = \text{ord}_2(k_1 + i - j_0) \leq 2 \log k_2 \quad (66)$$

for $1 \leq i \leq k_2 - k_1$. Then, we see from (63) and (66) that $l_2 - l_1 \leq 5 \log k_2$. This proves (64). Now, we combine (60), (62), (64) and (61) to derive that either $k_1 \leq c_{36}$ or $k_2 - k_1 \leq c_{36}$ which is not possible if $c_{30} > c_{36}$.

(b) Suppose that (1) and (2) with $k = k_1$, $l = l_1$ and $k = k_2$, $l = l_2$ are satisfied. Then, we observe that (64) is valid. Now, we apply Theorem 6(b) to derive that either $l_1 \leq c_{37} \log \log x$ or $l_2 - l_1 \leq c_{37} \log \log x$ where c_{37} is an effectively computable number depending only on ε , a and b . Finally, we observe that for a given l there is at most one k satisfying (2) to complete the proof of Theorem 2(b). $\square$

4.

Proof of Theorem 3(a). We denote by $d_1, \dots, d_5$ effectively computable positive numbers depending only on ε , a , b and $P(x)$. We may assume that $x > d_1$ with d_1 sufficiently large, otherwise the assertion follows from (1). By (24), (25) and (26), we see that

$$l \leq d_2 \max_{1 \leq i \leq k} \text{ord}_2(x + i) \leq d_2 \left(\max_{1 \leq i \leq k} \text{ord}_2(-xi^{-1} - 1) + \frac{\log k}{\log 2} \right).$$

Further, we write x as $\prod_{j=1}^{\omega(x)} p_j^{\text{ord}_{p_j}(x)}$ where $p_1, \dots, p_{\omega(x)}$ are the distinct prime factors of x . Now, we apply Lemma 10 to $\text{ord}_2(-xi^{-1} - 1)$ with $p = 2$, $n = \omega(x) + 2 \leq P(x) + 2$, $\alpha_1 = -1$, $b_1 = 1$; $\alpha_{j+1} = p_j$, $b_{j+1} = \text{ord}_{p_j}(x)$ with $1 \leq j \leq \omega(x)$ and $\alpha_n = i$, $b_n = -1$, to obtain

$$l \leq d_3 (\log \log x) \log k$$

which, together with (46) and (13), implies that

$$(\log k)(\log \log k)^2 \leq d_4 (\log x)(\log \log x).$$

Therefore, since $\omega(x) \leq P(x)$, we have

$$x > k^{\omega(x)}. \quad (67)$$

Consequently, there exists a prime p dividing x such that

$$p^{\text{ord}_p(x)} > k. \quad (68)$$

Now, we count the power of p on both the sides of (24) to derive that

$$[l/p] - 1 \leq \text{ord}_p(a) \quad (69)$$

which implies that $l \leq d_5$. $\square$

(b) We apply Lemma 8(a) with $\delta = 1 - \varepsilon$ to conclude that $k \leq d_6 l$ where d_6 and the subsequent letter d_7 are effectively computable numbers depending only on ε, a, b and $P(x)$. Now, we apply Theorem 3(a) to conclude that $\max(k, l) \leq d_7$. $\square$

(c) We write $d_8, \dots, d_{13}$ for effectively computable numbers depending only on a and b . There is no loss of generality in assuming that $x > d_8$ with d_8 sufficiently large. Suppose that (2) with $P(x) \leq l + k$ and $y \leq x/2$ is satisfied. For a prime p with $k < p \leq k + l$ and $p | x$, we derive from (2) that $p | a$. Therefore

$$P(x) \leq \max(k, P(a)).$$

Consequently, we observe from prime number theory that

$$\omega(x) \leq d_9 k / \log k. \quad (70)$$

First, we show that

$$l \leq d_{10} k. \quad (71)$$

Let $x \leq k^{\omega(x)}$. Then, we see from (23) and (70) that $l \leq d_{11} k$. Thus, we may assume (67) which implies (68), (69) and hence, $l \leq d_{12} k$. This proves (71). Finally, we combine (30), (71) and Theorem 6(b) with $\varepsilon = 1/2$ to conclude that $y > x^{d_{13}}$.

(d) Let $d_{14}, \dots, d_{18}$ be effectively computable positive numbers depending only on a, b and $P(x - y)$. We may assume that $x > d_{14}$ with d_{14} sufficiently large. Suppose that (2) with $y < x/2$ is satisfied. Then, the inequality (16) is valid. Further, we may suppose that

$$l \geq \max(P(x - y), 2a), \quad (72)$$

otherwise, we may derive from (30) and (16) that $y \geq x^{d_{15}}$.

Let $F(z)$ be given by (19). We re-write (2) as

$$0 \neq aF(x) - aF(y) = F(y)(b(y + k + 1) \cdots (y + k + l) - a). \quad (73)$$

For a prime p dividing $x - y$, we see from (73) and (72) that

$$\text{ord}_p(x - y) \leq \text{ord}_p(F(y)) + \text{ord}_p(a). \quad (74)$$

Further, we observe that

$$\text{ord}_p(F(y)) \leq \max_{1 \leq i \leq k} \text{ord}_p(y + i) + \left[\frac{k}{p} \right] + \left[\frac{k}{p^2} \right] + \cdots \quad (75)$$

Now, we derive from $y < x/2$, (74) and (75) that

$$\log \left(\frac{x}{2} \right) \leq \log(x - y) = \sum_{p | (x - y)} \text{ord}_p(x - y) \log p \leq d_{16} (\log(y + k) + k)$$

which, by (16) and (23), implies that either $y \geq x^{d_{17}}$ or $l \leq d_{18} k$. Finally, as above, we

$$\log\left(\frac{x}{2}\right) \leq \log(x-y) \leq \log a + \pi(l) \log(y+k) + 2k \sum_{p \leq l} \frac{\log p}{p}$$

$$\leq d_{18} \left(\frac{l \log k}{\log l} + k \log l \right)$$

which, together with (23) and Theorem 6(c), completes the proof. $\square$

Proof of Theorem 5. We denote by $d_{19}, \dots, d_{28}$ effectively computable positive numbers depending only on $a, b, P(x)$ and $P(y)$. We may assume that $x > d_{19}$ with d_{19} sufficiently large, otherwise the theorem follows from (1). We apply Lemma 9 to conclude that

$$x - y \geq x (\log x)^{-d_{20}}. \quad (76)$$

On the other hand, we derive from (46) and (23) that

$$x - y \leq d_{21} x (\log x)^2 / k. \quad (77)$$

We combine (76) and (77) to derive that

$$k \leq (\log x)^{d_{22}}. \quad (78)$$

Now, we show that

$$y \leq (\log x)^{d_{23}}. \quad (79)$$

For proving (79), we refer to (23) and (78) to assume that

$$y > (k + l)^4. \quad (80)$$

Then, by (27), (2), (15) and (80), we observe that

$$0 < U_k \leq d_{24} ((k + l)^2 y^{k+l-1} + k^2 x^{k-1}) \quad (81)$$

where U_k is given by (17). On the other hand, we apply again Lemma 9 to obtain

$$U_k \geq \max(x^k, y^{k+l}) ((k + l) \log x)^{-d_{25}}$$

which, together with (78) and (23), implies that

$$U_k \geq \max(x^k, y^{k+l}) (\log x)^{-d_{26}}. \quad (82)$$

Finally, we combine (81), (82), (78) and (80) to obtain (79). Therefore, we conclude from Theorem 3(b) with $\varepsilon = 1/2$ to obtain that $\max(k, l) \leq d_{27}$ which, together with (30) and (79), implies that $x = \max(x, y, k, l) \leq d_{28}$. $\square$

Proof of Corollary 1. We denote by d_{29}, d_{30} and d_{31} effectively computable positive numbers depending only on B . We may assume that $N \geq d_{29}$ with d_{29} sufficiently large. In (2), we put $a = 1$, $x = X$, $k = M$, $b = B!$, $y = B$ and $l = N - M - B$. By (6), we notice that $X \geq N$ so that (1) is satisfied. Therefore, we derive from Theorem 6(c) that

$$M \leq d_{30} \log \log X. \quad (83)$$

Further, we see from (5) that

$$N \geq d_{31}(\log X)/\log \log X. \quad (84)$$

Now, we apply Theorem 3(e), (83) and (84) to conclude that

$$P = P(X - B) > N - M - B > M + B. \quad (85)$$

Further, we see from (5) and (85) that

$$P \left| \left(\frac{N!}{M!} - \frac{(B+1) \cdots (B+M)}{M!} \right) \right|. \quad (86)$$

Finally, we derive from (86) and (85) that $P > N$. $\square$

5.

This section is devoted to preliminaries for the proof of Theorem 4. Let $F(z)$ be given by (19). Then

$$A_j = \sum_{i=i_0=1}^k \sum_{i_1=1}^{i_0-1} \cdots \sum_{i_{j-1}=1}^{i_{j-2}-1} i_0 i_1 \cdots i_{j-1} \quad \text{for } 1 \leq j \leq k.$$

We write the right hand side of (14)

$$(y+1) \cdots (y+2k) = \prod_{j=1}^k (u+j(2k-j+1)) \quad (87)$$

where

$$u = y(y+2k+1). \quad (88)$$

Let

$$G(z) = \prod_{j=1}^k (z+j(2k-j+1)) = z^k + B_1 z^{k-1} + \cdots + B_k \quad (89)$$

where

$$B_j = \sum_{i=i_0=1}^k \sum_{i_1=1}^{i_0-1} \cdots \sum_{i_{j-1}=1}^{i_{j-2}-1} i_0 i_1 \cdots i_{j-1} (2k-i_0+1) \cdots (2k-i_{j-1}+1) \quad (90)$$

for $1 \leq j \leq k$.

Further, we put

$$\Delta_q = \Delta_{q,k} = \sum_{i=1}^k i^q \quad \text{for } q = 1, 2, \dots$$

Then, we have $A_1 = \Delta_1$, $A_2 = (\Delta_3 - \Delta_2)/2$,

$$A_3 = (3\Delta_5 - 10\Delta_4 + 9\Delta_3 - 2\Delta_2)/24,$$

and

$$\begin{aligned} 3B_3 = & -\frac{1}{6}\Delta_8 + \left(\frac{4}{3}k + \frac{22}{15}\right)\Delta_7 - \left(\frac{7}{2}k^2 + \frac{91}{10}k + \frac{149}{30}\right)\Delta_6 \\ & + \left(3k^3 + \frac{35}{2}k^2 + \frac{137}{6}k + \frac{49}{6}\right)\Delta_5 - \left(10k^3 + \frac{59}{2}k^2 + \frac{51}{2}k + \frac{20}{3}\right)\Delta_4 \\ & + \left(9k^3 + \frac{37}{2}k^2 + \frac{71}{6}k + \frac{71}{30}\right)\Delta_3 - \left(2k^3 + 3k^2 + \frac{7}{5}k + \frac{1}{5}\right)\Delta_2. \end{aligned}$$

Further, we derive from [9, p. 6] that

$$\begin{aligned} \Delta_1 &= k(k+1)/2, \quad \Delta_2 = k(k+1)(2k+1)/6, \quad \Delta_3 = k^2(k+1)^2/4, \\ \Delta_4 &= k(k+1)(6k^3+9k^2+k-1)/30, \quad \Delta_5 = k^2(k+1)(2k^3+4k^2+k-1)/12, \\ \Delta_6 &= k(k+1)(6k^5+15k^4+6k^3-6k^2-k+1)/42, \\ \Delta_7 &= k^2(k+1)(3k^5+9k^4+5k^3-5k^2-2k+2)/24 \end{aligned}$$

and

$$\Delta_8 = k(k+1)(10k^7+35k^6+25k^5-25k^4-17k^3+17k^2+3k-3)/90.$$

Consequently, we obtain the following expressions for A_1, A_2, A_3, A_4 and B_1, B_2, B_3 .

Lemma 11. $A_1 = k(k+1)/2, A_2 = k(k+1)(k-1)(3k+2)/24,$

$$A_3 = k^2(k+1)^2(k-1)(k-2)/48,$$

$$A_4 = k(k+1)(k-1)(k-2)(k-3)(15k^3+15k^2-10k-8)/5760 \quad (91)$$

and

$$B_1 = k(k+1)(2k+1)/3, \quad B_2 = k(k+1)(20k^4+16k^3-11k^2-19k-6)/90, \quad (92)$$

$$\begin{aligned} B_3 &= k(k+1)(280k^7-28k^6-830k^5-745k^4 \\ &+ 136k^3+557k^2+486k+144)/5670. \end{aligned}$$

Let

$$f = \left\lceil \frac{2k^2}{3} \right\rceil + k - \left\lceil \frac{k}{2} \right\rceil.$$

We write

$$F(z+f) = z^k + A_1(f)z^{k-1} + \dots + A_k(f) \quad (93)$$

where

$$A_i(f) = \binom{k}{i} f^i + \binom{k-1}{i-1} f^{i-1} A_1 + \dots + \binom{k-i+1}{1} f A_{i-1} + A_i \quad (94)$$

for $1 \leq i \leq k$.

Then, we apply Lemma 11 to obtain the following result.

Lemma 12. Let

$$\delta = \begin{cases} 0 & \text{if } k \equiv 0 \pmod{6} \\ -1/6 & \text{if } k \equiv 1, 5 \pmod{6} \\ -2/3 & \text{if } k \equiv 2, 4 \pmod{6} \\ 1/2 & \text{if } k \equiv 3 \pmod{6} \end{cases} \quad (95)$$

Then

$$A_1(f) - B_1 = \left(\frac{1}{6} + \delta\right)k,$$

$$A_2(f) - B_2 = \frac{2}{45}k^5 + \left(\frac{1}{9} + \frac{2\delta}{3}\right)k^4 + \left(-\frac{1}{24} + \frac{\delta}{3}\right)k^3 + \left(-\frac{7}{72} + \frac{\delta}{2}(\delta - 1)\right)k^2 \\ + \left(-\frac{1}{60} - \frac{\delta}{2}(1 + \delta)\right)k$$

and

$$A_3(f) - B_3 = \frac{4}{135}k^8 + \left(\frac{79}{2835} + \frac{2}{9}\delta\right)k^7 - \frac{5}{36}k^6 + \left(-\frac{169}{1080} - \frac{13}{18}\delta + \frac{1}{3}\delta^2\right)k^5 \\ + \left(\frac{67}{720} - \frac{17}{24}\delta - \frac{1}{2}\delta^2\right)k^4 + \left(\frac{1013}{6480} + \frac{9}{24}\delta - \frac{7}{12}\delta^2 + \frac{1}{6}\delta^3\right)k^3 \\ + \left(\frac{1}{72} + \frac{2}{3}\delta + \frac{1}{4}\delta^2 - \frac{1}{2}\delta^3\right)k^2 \\ + \left(-\frac{8}{315} + \frac{1}{6}\delta + \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3\right)k.$$

Proof. We write $g = \frac{2}{3}k^2 + \frac{1}{2}k$ and $f - g = \delta$. Then, we check that (95) is satisfied.

Further, we see from (94) that $A_1(f) = kg + A_1 + \delta k$, $A_2(f) = \binom{k}{2}g^2 + (k-1)A_1g + A_2 + \delta \binom{k}{2}(2g + \delta) + \delta(k-1)A_1$ and $A_3(f) = \binom{k}{3}g^3 + \binom{k-1}{2}A_1g^2 + \binom{k-2}{1}A_2g + A_3 + \delta \binom{k}{3}(3g^2 + 3\delta g + \delta^2) + \delta \binom{k-1}{2}A_1(2g + \delta) + \delta \binom{k-2}{1}A_2$. Now, we apply Lemma 11 to complete the proof of Lemma 12. $\square$

Finally, we obtain estimates for all $A_i(f)$ and B_j .

Lemma 13. For $1 \leq i \leq k$, we have

$$A_i(f) \leq (1 + \phi_k) \binom{k}{i} f^i \quad (96)$$

where

$$\phi_k = \frac{3}{4} \left(1 + \frac{1}{k}\right)^2 \left(1 - \frac{3}{8} \left(1 + \frac{1}{k}\right)^2\right)^{-1}. \quad (97)$$

Furthermore,

$$B_j \leq k!(k+1)^{2j}/j! \quad \text{for } 1 \leq j \leq k \quad (98)$$

and

$$B_4 \leq (2k)^4 A_4, \quad B_4 \geq (k+1)^4 A_4. \quad (99)$$

Proof. By (94), we see that

$$A_j f^{-j} \leq \frac{1}{j!} \left(\frac{3}{4}\right)^j \left(1 + \frac{1}{k}\right)^{2j}$$

which, together with (100), implies (96) where ϕ_k is given by (97). By (90), we see that $B_4 \geq (k+1)^4 A_4$ and

$$B_j \leq (2k)^j A_j, \quad \text{for } 1 \leq j \leq k,$$

which, by (21), implies (98). □

6.

In this section, we shall prove Theorem 4. The computations for the proof of Theorem 4 are carried out on a pocket calculator. Throughout this section, we assume that (14) is satisfied. For a prime $p > 0$, we write

$$n = b_0 + b_1 p + \dots + b_\mu p^\mu$$

where $b_0, \dots, b_\mu$ are integers satisfying $b_\mu \neq 0$ and $0 \leq b_i < p$ for $0 \leq i \leq \mu$. Then, we start with the following well-known result.

Lemma 14.

$$\text{ord}_p(n!) = \frac{n - (b_0 + \dots + b_\mu)}{p - 1}. \quad (101)$$

For the proof of Lemma 14, we observe that

$$\text{ord}_p(n!) = \sum_{i=1}^{\mu} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{\mu} (b_i + b_{i+1}p + \dots + b_\mu p^{\mu-i})$$

which implies (101). We denote by τ the number of positive integers i such that

$$\left\lfloor \frac{y+2k}{2^i} \right\rfloor - \left\lfloor \frac{y}{2^i} \right\rfloor - \left\lfloor \frac{2k}{2^i} \right\rfloor > 0.$$

Then, we apply Lemma 14 to obtain the following lower bound for x and this is fundamental for our argument.

Lemma 15.

$$x \geq 2^{k+\tau} - k. \quad (102)$$

Since $\tau \geq 0$, the inequality (102) implies that

$$x \geq 2^k - k. \quad (103)$$

Proof. We re-write (14) as

$$\frac{(x+1) \cdots (x+k)}{k!} = \frac{(y+1) \cdots (y+2k)(2k)!}{(2k)! k!}. \quad (104)$$

By (101),

$$\text{ord}_2\left(\frac{(2k)!}{k!}\right) = k. \quad (105)$$

Further, we observe that

$$\text{ord}_2\left(\frac{(y+1)\cdots(y+2k)}{(2k)!}\right) = \sum_{i=1}^{\infty} \left(\left\lfloor \frac{y+2k}{2^i} \right\rfloor - \left\lfloor \frac{y}{2^i} \right\rfloor - \left\lfloor \frac{2k}{2^i} \right\rfloor \right)$$

and every summand is either 0 or 1. Therefore

$$\text{ord}_2\left(\frac{(y+1)\cdots(y+2k)}{(2k)!}\right) = \tau. \quad (106)$$

On the other hand, we see that

$$\text{ord}_2\left(\frac{(x+1)\cdots(x+k)}{k!}\right) \leq \max_{1 \leq i \leq k} \text{ord}_2(x+i) \leq (\log(x+k))/\log 2. \quad (107)$$

Finally, we combine (104), (105), (106) and (107) to obtain (102).

As an application of Lemma 15, we prove

Lemma 16. If (14) holds, then

$$(x+1)\cdots(x+k) > \left(x + \frac{k}{2}\right)^k. \quad (108)$$

Proof. It is easy to check (108) for $k \leq 4$. Thus, we may assume that $k \geq 5$. Suppose that

$$(x+1)\cdots(x+k) \leq \left(x + \frac{k}{2}\right)^k. \quad (109)$$

By Lemma 11, we observe that $A_1 - \binom{k}{1}\left(\frac{k}{2}\right) = \frac{k}{2}$, $A_2 - \binom{k}{2}\left(\frac{k}{2}\right)^2 = \frac{k(k-1)(5k+2)}{24}$

and $A_3 - \binom{k}{3}\left(\frac{k}{2}\right)^3 = \frac{k^2(k-1)(k-2)(2k+1)}{48}$ are positive. Further, we see from (103) and $k \geq 5$ that $x > k^2$. Now, we derive from (109) that

$$\frac{1}{2}kx^{k-1} \leq \left(\binom{k}{4}\left(\frac{k}{2}\right)^4\right)x^{k-4}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) \leq 5k^8x^{k-4}/1728$$

which implies that $172x^3 < k^7$. Now, since $x > k^2$, we derive that $k > 172$. Then, we apply (103) to conclude that $x > k^3$ and this is a contradiction.

For small values of y and k , the next two lemmas are useful. The first may be confirmed by direct checking.

and

$$P((x+1)\cdots(x+6)) \geq 37 \quad \text{for } 120 \leq x \leq 600.$$

Lemma 18. Equation (14) implies that

$$y \geq \begin{cases} 17 & \text{if } k \in \{5, 6, 7, 8\} \\ 28 & \text{if } k = 9 \\ 32 & \text{if } k = 10 \end{cases}$$

Proof. By the arithmetic-geometric mean, we observe that

$$((x+1)\cdots(x+k))^{1/k} < \frac{(x+1) + \cdots + (x+k)}{k} = x + \frac{k+1}{2}$$

which, together with (108), implies that

$$[(x+1)\cdots(x+k)]^{1/k} = x + \left\lfloor \frac{k}{2} \right\rfloor. \quad (110)$$

By applying again the arithmetic-geometric mean, we see from (87) and (92) that

$$[(y+1)\cdots(y+2k)]^{1/k} \leq u + \left\lceil \frac{(k+1)(2k+1)}{3} \right\rceil. \quad (111)$$

Now, we combine (14), (110) and (111) to derive that

$$0 < x - u \leq \left\lceil \frac{(k+1)(2k+1)}{3} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor. \quad (112)$$

Let $k \in \{5, 6, 7, 8\}$ and $y \leq 16$. Then, we see from (88) and (112) that $x \leq 600$. Now, we apply Lemma 17 to derive that $x < 54$ if $k \in \{5, 6, 7\}$ and $x < 120$ if $k = 8$ which, together with (103), imply that $k = 5$ and $x \geq 27$. Therefore, we see from (88) and (112) that $y(y+1) < 54$ which implies that $y \leq 3$. Furthermore, we observe $\lceil (y+10)/4 \rceil - \lfloor y/4 \rfloor - \lfloor 10/4 \rfloor = 1$ if $y = 2, 3$ to derive from Lemma 15 with $\tau \geq 1$ that $y \leq 1$. Then, since the left hand side of (14) is not divisible by 31, we see that $x \geq 31$. Consequently, we observe that the left hand side of (14) is greater than the right hand side of (14).

Let $k = 10$ and $y \leq 31$. Then, we observe from (88) and (112) that $x \leq 1684$ which, by Lemma 15, implies that $\tau = 0$. On the other hand, we observe that $\lceil (y+20)/32 \rceil - \lfloor y/32 \rfloor - \lfloor 20/32 \rfloor = 1$ if $y \geq 12$. Consequently, we conclude that $y \leq 11$ which implies that $x \leq 424$ contradicting (103).

Let $k = 9$ and $y \leq 27$. Then, by (88), (112) and (103), we observe that $x \leq 1301$, $x \geq 503$ and $y \geq 14$. Then, we observe $\lceil (y+18)/32 \rceil - \lfloor y/32 \rfloor - \lfloor 18/32 \rfloor = 1$ to apply Lemma 15 with $\tau \geq 1$ to derive that $x \geq 1015$ which implies that $y > 22$. If $y \in \{23, 26, 27\}$, we see $\tau \geq 2$ to obtain from (102) that $x \geq 2039$ which is not possible. If $y = 24$, then $x \leq 1091$ and, by looking at the prime factors of 1086, 1091, 1093, we derive from (14) that $x \leq 1076$ which is not possible, since the left hand side of (14) is less than the right hand side of (14). If $y = 25$, then $x \leq 1159$ and we argue, as above by looking at the prime factors of 1151, 1159, 1162, to derive $x \leq 1149$ for arriving

Now, we are ready to prove an inequality for (87) analogous to (108).

Lemma 19. Let $u > 0$ if $k = 3$ and

$$\theta = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{3} \\ 2 & \text{if } k \equiv 1, 2 \pmod{3} \end{cases}$$

Then

$$\prod_{j=1}^k (u + j(2k - j + 1)) > \left(u + \frac{(k+1)(2k+1) - \theta}{3} \right)^k.$$

Proof. The Lemma can be verified for $k \leq 4$. By direct computations, we may assume that $u \leq 21$ if $k = 5$, $u \leq 145$ if $k = 6$ and $u \leq 125$ if $k = 7$. Now, we refer to (88) and Lemma 18 to exclude these possibilities. Thus, we may suppose that $k \geq 8$. We set

$$h(\theta) = h(\theta, k) = ((k+1)(2k+1) - \theta)/3.$$

We observe that

$$h(\theta) \leq k(2k+3)/3. \quad (113)$$

We suppose that

$$\prod_{j=1}^k (u + j(2k - j + 1)) \leq (u + h(\theta))^k. \quad (114)$$

By (113), we see that

$$\begin{aligned} (u + h(\theta))^k &\leq u^k + kh(\theta)u^{k-1} \\ &\quad + \binom{k}{2} \left(\frac{k(2k+3)}{3} \right)^2 u^{k-2} + \dots + \binom{k}{k} \left(\frac{k(2k+3)}{3} \right)^k. \end{aligned} \quad (115)$$

We put

$$D_2 = \frac{2}{45}k^5 - \frac{2}{9}k^4 - \frac{1}{6}k^3 + \frac{5}{18}k^2 + \frac{1}{15}k,$$

$$D_3 = \frac{k(k-1)(k-2)}{5670} (168k^5 + 32k^4 - 870k^3 - 1120k^2 - 423k - 72)$$

and

$$R' = \binom{k}{5} \left(\frac{k(2k+3)}{3} \right)^5 u^{k-5} + \dots + \binom{k}{k} \left(\frac{k(2k+3)}{3} \right)^k.$$

Further, we observe from (112), (103) and $k \geq 8$ that $u > k^2(2k+3)/9$. Therefore

$$R' \leq 2 \binom{k}{5} \left(\frac{k(2k+3)}{3} \right)^5 u^{k-5}. \quad (116)$$

Thus, we derive from (114), (89), (115), (116) and Lemma 11 that

$$\begin{aligned} \left(\frac{\theta k}{3} \right) u^{k-1} &\leq D_2 u^{k-2} + D_3 u^{k-3} + \binom{k}{4} \left(\frac{k(2k+3)}{3} \right)^4 u^{k-4} \\ &\quad + 2 \binom{k}{5} \left(\frac{k(2k+3)}{3} \right)^5 u^{k-5}. \end{aligned} \quad (117)$$

We derive from (117) that $u \leq 550$ if $k = 8$, $u \leq 1200$ if $k = 9$ and $u \leq 1500$ whenever $k = 10$. Now, we apply (88) and Lemma 18 to exclude these cases. Furthermore, the inequality (117) implies that $u < 1950$ if $k = 11$, $u < 3980$ if $k = 12$ and these cases are excluded by (112) and (103).

Thus, we may assume that $k \geq 13$. Then, we see from (112) and (103) that

$$u \geq \frac{7}{25}k^4. \quad (118)$$

Further, we observe that

$$D_2 < \frac{2}{45}k^5, \quad D_3 < \frac{171}{5670}k^8,$$

$$\binom{k}{4} \left(\frac{k(2k+3)}{3} \right)^4 \leq \frac{1}{24} \left(\frac{29}{39} \right)^4 k^{12} \leq \frac{13}{1000} k^{12}$$

and

$$2 \binom{k}{5} \left(\frac{k(2k+3)}{3} \right)^5 \leq \frac{1}{60} \left(\frac{29}{39} \right)^5 k^{15} \leq \frac{1}{250} k^{15}.$$

Therefore, we derive from (117) and $\theta \geq 1$ that

$$u < \frac{2}{15}k^4 + \frac{513}{5670} \frac{k^7}{u} + \frac{39}{1000} \frac{k^{11}}{u^2} + \frac{3}{250} \frac{k^{14}}{u^3}$$

which, together with (118), implies that

$$\frac{7}{25}k^4 \leq u < \frac{2}{15}k^4 + k^3 + k^2.$$

Consequently, we conclude that $k \leq 7$ which is a contradiction. $\square$

As an immediate consequence of Lemma 19 and (110), we obtain the following improvement of (112).

COROLLARY 3.

Suppose that (14) is satisfied. Then, either $x = 7$, $y = 0$, $k = 3$ or

$$x - u = f. \quad (119)$$

Proof. We may assume that $u > 0$ if $k = 3$. Then, we see from Lemma 19 that

$$\left[\left(\prod_{j=1}^k (u + j(2k - j + 1)) \right)^{1/k} \right] \geq u + [h(\theta)] = u + \left[\frac{k(2k+3)}{3} \right]. \quad (120)$$

On the other hand, we see from the arithmetic-geometric mean and (92) that the left hand side of (120) is less than $u + [(k+1)(2k+1)/3]$. Consequently, we conclude that (120) holds with equality sign. Hence, the assertion (119) follows from (14), (87)

(87) and (119) that

$$(u+f+1)\cdots(u+f+k) = \prod_{j=1}^k (u+j(2k-j+1)). \quad (121)$$

We may verify, by direct computation, that (121) is not possible whenever $k \leq 7$. Thus, we may suppose that $k \geq 8$. Further, by (121), (93) and (89), we see that

$$(A_1(f) - B_1)u^{k-1} + (A_2(f) - B_2)u^{k-2} + \cdots + (A_k(f) - B_k) = 0.$$

Thus

$$\begin{aligned} & (A_1(f) - B_1)u^{k-1} + (A_2(f) - B_2)u^{k-2} + (A_3(f) - B_3)u^{k-3} \\ & \leq (B_4 - A_4(f))u^{k-4} + R_3 \end{aligned} \quad (122)$$

and

$$\begin{aligned} & (B_1 - A_1(f))u^{k-1} + (B_2 - A_2(f))u^{k-2} + (B_3 - A_3(f))u^{k-3} \\ & \leq (A_4(f) - B_4)u^{k-4} + R_4 \end{aligned} \quad (123)$$

where

$$R_3 = B_5u^{k-5} + \cdots + B_k, \quad R_4 = A_5(f)u^{k-5} + \cdots + A_k(f). \quad (124)$$

By (119) and (103), we see that

$$u \geq \frac{22}{45}k(k+1)^2 \text{ if } k \geq 9 \quad \text{and} \quad u \geq \frac{1}{2}fk. \quad (125)$$

Thus, we derive from (124), (96), (98) and (125) that

$$R_3 \leq \frac{2}{145}k^5(k+1)^{10}u^{k-5} \quad \text{if } k \geq 9 \quad \text{and} \quad R_4 \leq \frac{17}{4}f^5 \binom{k}{5}u^{k-5}. \quad (126)$$

Next, we turn to the coefficient $B_4 - A_4(f)$ in (122) and (123). By (91), we calculate $A_4 = 22449$ if $k = 8$ and $A_4 = 157773$ if $k = 10$. Then, we derive from (94) and (99) that

$$(A_4(f) - B_4)/10^4 \leq \begin{cases} 30730 & \text{if } k = 8 \\ 487570 & \text{if } k = 10 \end{cases} \quad (127)$$

Similarly, by observing $B_4 - A_4(f) \leq B_4 - \binom{k}{4}f^4 - \binom{k-1}{3}A_1f^3$, we obtain

$$(B_4 - A_4(f))/10^5 \leq \begin{cases} 45990 & \text{if } k = 9 \\ 606420 & \text{if } k = 11 \end{cases} \quad (128)$$

and

$$(B_4 - A_4(f))/u^2 \leq \begin{cases} 11450 & \text{if } k = 12 \\ 7660 & \text{if } k = 13 \end{cases} \quad (129)$$

since, by (119) and (103), $u \geq 3982$ if $k = 12$ and $u \geq 8060$ if $k = 13$. Next, we notice that

$$f \leq \frac{3}{4}k^2 \quad (130)$$

and

$$u \geq \frac{2}{3}k^4 \quad \text{if } k \geq 14 \quad (131)$$

$$A_2(f) - B_2 < \frac{2}{45}k^5 - \frac{1}{3}k^4, \quad A_3(f) - B_3 < \frac{4}{135}k^8 - \frac{341}{2835}k^7,$$

since $k \geq 8$. Therefore, we derive from (123) that

$$\begin{aligned} \frac{1}{2}ku^{k-1} &< \left(\frac{2}{45}k^5 - \frac{1}{3}k^4\right)u^{k-2} + \left(\frac{4}{135}k^8 - \frac{341}{2835}k^7\right)u^{k-3} \\ &+ (A_4(f) - B_4)u^{k-4} + R_4. \end{aligned} \quad (132)$$

Then, we see from (132), (127) and (126) that $u \leq 530$ if $k = 8$ and $u \leq 1300$ if $k = 10$. Now, we apply (88) and Lemma 18 to assume that $k \geq 14$. Further, we see from (96), (97), (130) and (126) that

$$A_4(f) \leq (1 + \phi_{14})f^4k^4/24 \leq k^{12}/25 \quad (133)$$

and

$$R_4 \leq 17f^5k^5u^{k-5}/480 \leq 9k^{15}u^{k-5}/1000. \quad (134)$$

Thus, we combine (132), (131), (133) and (134) to derive that

$$\frac{2}{5}k^4 \leq u < \frac{4}{45}k^4 + \frac{8}{135}\frac{k^7}{u} + \frac{2}{25}\frac{k^{11}}{u^2} + \frac{9}{500}\frac{k^{14}}{u^3} \leq \frac{4}{45}k^4 + k^3 + k^2$$

which implies that $k \leq 4$.

We argue, as above, to derive from (122) that

$$\frac{2}{3}ku^{k-1} \leq (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 3 \pmod{6}), \quad (135)$$

$$\begin{aligned} \frac{1}{6}ku^{k-1} + \left(\frac{2}{45}k^5 + \frac{7}{72}k^4\right)u^{k-2} \\ \leq (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 0 \pmod{6}) \end{aligned} \quad (136)$$

and

$$\left(\frac{2}{45}k^5 - \frac{7}{72}k^3\right)u^{k-2} \leq (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 1, 5 \pmod{6}). \quad (137)$$

We see from (136), (137), (129) and (126) that $u < 3982$ if $k = 12$ and $u < 8060$ if $k = 13$. On the other hand, we apply (119) and (103) to exclude these possibilities. Further, we see from (135), (137), (128) and (126) that $u \leq 1300$ if $k = 9$ and $u \leq 3950$ if $k = 11$. In view of (88) and Lemma 18, the first case is not possible. Let $k = 11$. Then $\tau = 0$ and $35 \leq y \leq 52$. Further, we observe $[(y+22)/64] - [y/64] - [22/64] = 1$ if $y \geq 42$ to derive from Lemma 15 that $y \leq 41$. Further, corresponding to every y with $35 \leq y \leq 41$, there is precisely one value of x given by (119) and (88). Finally, we count the power of 2 on both the sides of (14) in each of these seven cases to arrive at a contradiction.

Thus, we may assume that $k \not\equiv 2, 4 \pmod{6}$ and $k \geq 15$. Then, we apply (131) to derive from (135), (136) and (137) that

$$\left(\frac{2}{45}k^5 - \frac{7}{72}k^3\right)u^{k-2} \leq (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \not\equiv 2, 4 \pmod{6}). \quad (138)$$

Further, by (98) and (126), we see that

$$B_4 - A_4(f) < B_4 \leq 3k^{12}/40, \quad R_3 \leq 4k^{15}u^{k-5}/145. \quad (139)$$

Now, we combine (138), (139) and (131) to obtain

$$\frac{2}{45}k^5 \leq \frac{3}{40} \frac{k^{12}}{u^2} + \frac{4}{145} \frac{k^{15}}{u^3} + \frac{7}{72}k^3 \leq \frac{15}{32}k^4 + k^3$$

which implies that $k \leq 12$. □

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Extension of certain types of generating relations

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Abstract. This paper gives a new expansion formula which essentially involves a double sum. As a consequence of our main result, eq. (6), various types of generating relations are seen to emerge and relevance with some known results is pointed out briefly. The usefulness of our main result is also indicated by considering its application to a probabilistic method for a lattice path enumeration problem.

Keywords. Generating relations; combinatorial coefficients; Laguerre polynomials; lattice path enumeration.

1. Introduction and the main result

For any real or complex λ , the binomial (or combinatorial) coefficients are defined by

$$\binom{\lambda}{n} = (-1)^n (-\lambda)_n / n!, \quad n \in \{0, 1, 2, \dots\}, \quad (1)$$

where $(a)_n$ denotes the usual Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n=0, \\ a(a+1)\cdots(a+n-1), & \forall n \in \mathcal{N}. \end{cases} \quad (2)$$

In his paper Carlitz [1] derived a generating function for the Laguerre polynomials (see also [8, p. 378])

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta n)}(x+ny)t^n = \frac{\exp(-xw)(1+w)^{\alpha+1}}{1-\beta w+yw(1+w)}, \quad (3)$$

where w is implicitly defined in terms of t by

$$w = t \exp(-yw)(1+w)^{\beta+1}. \quad (4)$$

For subsequent extension of Carlitz's work, one may refer to Srivastava [7].

The objective of this paper is to obtain a new generalization of (3) in the light of Gould's important combinatorial relation [4, p. 196, eq. (6.1)] (see also [8, p. 356, eq. (13)]). Our main result is given below:

Theorem. For arbitrary parameters α , β and γ (independent of n and k), let

$$S_n(\alpha, \beta, \gamma; z) = \sum_{k=0}^n \binom{\alpha + (\beta+1)n}{n-k} z^k / k! (\gamma + (\beta+1)n). \quad (5)$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} S_n(\alpha, \beta, \gamma; x + yn)t^n \\
&= (1+w)^\alpha \exp[yw\gamma/(\beta+1)] \sum_{n,k=0}^{\infty} \binom{\alpha-\gamma}{n} \binom{n+k+\gamma/(\beta+1)}{n}^{-1} \\
&\quad \times (-w/(1+w))^n [w(x-y\gamma/(\beta+1))]^k/k! (\gamma+(\beta+1)k), \tag{6}
\end{aligned}$$

provided that $\operatorname{Re}[\gamma/(\beta+1)] > 0$, w is a function of t given by

$$w = t \exp(yw)(1+w)^{\beta+1}, \tag{7}$$

and the parameters assume such values that both sides of (6) exist.

2. Proof of (6)

Let us replace t by $t^{\beta+1}$ in an equivalent form of the Carlitz's formula (3) (see Raina [5, p. 592, eq. (2.3)]). Both sides of the resulting equation is multiplied by $t^{\gamma-1} dt$, then we get

$$\begin{aligned}
& \sum_{n,k=0}^{\infty} \binom{\alpha+(\beta+1)(n+k)}{n} [(x+y(n+k))^k/k!] t^{\gamma+(\beta+1)(n+k)-1} dt \\
&= \frac{e^{xw}(1+w)^{\alpha+1}}{1-\beta w-yw(1+w)} t^{\gamma-1} \frac{dt}{dw} \cdot dw, \tag{8}
\end{aligned}$$

where w is now given by

$$w = \exp(yw)[t(1+w)]^{\beta+1}. \tag{9}$$

On obtaining dt/dw from (9) by logarithmic differentiation, substituting it on the right side of (8), then on invoking (9) and integrating the resulting equation from (8) yields

$$\begin{aligned}
& \sum_{n,k=0}^{\infty} \binom{\alpha+(\beta+1)(n+k)}{n} [(x+y(n+k))^k/k!] \left[\frac{t^{\gamma+(\beta+1)(n+k)}}{\gamma+(\beta+1)(n+k)} \right]_0^t \\
&= (\beta+1)^{-1} \int_0^w w^{\gamma/(\beta+1)-1} (1+w)^{\alpha-\gamma} \exp[w(x-y\gamma/(\beta+1))] dw, \tag{10}
\end{aligned}$$

provided that $\operatorname{Re}[\gamma/(\beta+1)] > 0$.

By appealing to a special case of an integral given in [2, p. 71, eq. (2.11.4)] to evaluate the integral on the right side, and then making use of a linear transformation [8, p. 33, eq. (19)] for the ${}_2F_1$ -function involved, the resulting equation from (10) in accompaniment with (7) and the defining equation (5) is seen to lead to the desired result (6) upon setting t for $t^{\beta+1}$.

Remark 1. The formula (6) can also be expressed in the equivalent form

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} L_n^{(\alpha+\beta n)}(-x-ny)t^n \tag{11}$$

$$= (1+w)^x \exp[yw\gamma/(\beta+1)]$$

$$\Xi_1[\gamma - \alpha, \gamma/(\beta+1), 1; 1 + \gamma/(\beta+1); w/(1+w), w(x - y\gamma/(\beta+1))],$$

where w (as before) is given by (7), and the function $\Xi_1[x, y]$ is one of the Humbert's confluent hypergeometric function of two variables [8, p. 59, eq. (43)].

Remark 2. It may be observed that in the limiting case when $|\gamma| \rightarrow \infty$, (11) corresponds (formally) to the Carlitz's formula (3), a fact which can be verified.

3. Applications

The main result (6) can be specialized and applied to derive several classes of generating functions. Most of these results have relevance with the results recorded in Srivastava-Manocha monograph [8]. We leave to the interested reader to verify the outcome of such results from (6). We however point out the application of (6) in a certain probabilistic method in finding the solution of a functional equation involving generating functions for lattice path enumeration problem.

As explained in [3], let A_n denote the number of paths in a plane from the origin to the point $(n, \alpha + \beta n)$, where α and β are positive integers, with unit steps in the positive horizontal and vertical directions, which never touch the line $y = \alpha + \beta x$, except at the end point. If p is the probability that the particle starting from the origin moves one unit to the right and with probability $q = (1 - p)$ one unit up, then the probability that the particle stops at $(n, \alpha + \beta n)$ is $A_n p^n q^{\alpha + \beta n}$.

If we let

$$A(t) = \sum_{n=0}^{\infty} A_n t^n, \quad (12)$$

then for sufficiently small p , the particle will eventually reach the line $y = \alpha + \beta x$, with probability 1, satisfying

$$q^\alpha A(pq^\beta) = 1. \quad (13)$$

To solve the functional eq. (13), we use (6) in the following manner:

We set $y = \alpha$, and $x = y = 0$, in (6), and then putting $1 + w = q^{-1}$, in the resulting simplified form, so that $w = (1 - q)/q = p/q$, we then have

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} t^n / (\alpha + (\beta + 1)n) = q^{-\alpha}, \quad (14)$$

where

$$t = pq^\beta. \quad (15)$$

A comparison of (13) and (14) in view of (12) at once gives the number of paths A_n in the plane by

$$A_n = \frac{1}{\alpha + (\beta + 1)n} \binom{\alpha + (\beta + 1)n}{n}. \quad (16)$$

In a similar way, other simpler cases as investigated in [3, pp. 50–51] may be treated

with the help of our general formula (6). We conclude by remarking that the present work can be extended further on the lines of recent papers of Raina [6] and Srivastava and Raina [9].

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Some uncertainty inequalities

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Abstract. We prove an uncertainty inequality for the Fourier transform on the Heisenberg group analogous to the classical uncertainty inequality for the Euclidean Fourier transform. Inequalities of similar form are obtained for the Hermite and Laguerre expansions.

Keywords. Heisenberg group; Fourier transform; Hermite expansion; Laguerre expansion.

1. Introduction

The classical uncertainty inequality for the Fourier transform on $\mathbb{R}^n$ states that

$$\left(\int |x|^2 |f(x)|^2 dx \right) \left(\int |\xi|^2 |\hat{f}(\xi)|^2 d\xi \right) \geq \frac{n^2}{4} \quad (1)$$

for all normalized L^2 functions f . Here the Fourier transform of the function f is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int \exp(-ix \cdot \xi) f(x) dx. \quad (2)$$

The above inequality can be written in the form

$$\| |x| f \|_2 \| (-\Delta)^{1/2} f \|_2 \geq \frac{n}{2} \quad (3)$$

where $(-\Delta)^{1/2}$ is defined by the equation

$$((-\Delta)^{1/2} f)^\wedge(\xi) = |\xi| \hat{f}(\xi). \quad (4)$$

The purpose of this note is to prove similar inequalities when $-\Delta$ is replaced by some other differential operator.

First we consider the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. For normalized L^2 functions on $\mathbb{R}^n$ we prove that

$$\| |x| f \|_2 \| H^{1/2} f \|_2 \geq \frac{\sqrt{n}}{2}. \quad (5)$$

Here $H^{1/2} f$ is defined by

$$H^{1/2} f(x) = \sum (2N + n)^{1/2} P_N f(x) \quad (6)$$

$$f(x) = \sum P_N f(x).$$

For example when $n = 1$ we get the inequality

$$\left(\int t^2 |f(t)|^2 dt \right)^{1/2} (\sum (2k+1) |\hat{f}(k)|^2)^{1/2} \geq \frac{1}{2} \quad (7)$$

where $\hat{f}(k)$ are the Hermite coefficients of the function f .

Next we consider the operator L ,

$$L = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2) - 1 \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right) \quad (8)$$

on $\mathbb{C}^n$. This operator is connected with special Hermite expansions. For this operator we prove the inequality

$$\| |z| f \|_2 \| L^{1/2} f \|_2 \geq \frac{\sqrt{n}}{2} \quad (9)$$

for all normalized L^2 functions f on $\mathbb{C}^n$. If we consider only radial functions the above inequality becomes an uncertainty inequality for the Laguerre expansions. For example, when $n = 1$ we have

$$\left(\int_0^\infty t^2 |f(t)|^2 dt \right)^{1/2} \left(\sum_{k=0}^\infty (2k+1) |\hat{f}(k)|^2 \right)^{1/2} \geq \frac{\sqrt{n}}{2} \quad (10)$$

where $\hat{f}(k)$ are defined by

$$\hat{f}(k) = \int_0^\infty f(t) L_k(\tfrac{1}{2}t) \exp(-\tfrac{1}{4}t) dt \quad (11)$$

where $L_k(t)$ are the usual Laguerre polynomials.

Finally we consider the sublaplacian $\mathcal{L}$ on the Heisenberg group H^n . For normalized functions f on $L^2(H^n)$ we prove that

$$\begin{aligned} & \left(\iint |z|^2 |f(z, t)|^2 dz dt \right)^{1/2} \left(\iint |\mathcal{L}^{1/2} f(z, t)|^2 dz dt \right)^{1/2} \\ & \geq \sqrt{n} \left(\frac{\pi}{2} \right)^{(n+1)/2} \end{aligned} \quad (12)$$

The above inequality can also be written in the form

$$\begin{aligned} & \left(\iint |z|^2 |f(z, t)|^2 dz dt \right)^{1/2} \left(\int |\lambda|^n \|\hat{f}(\lambda) H(\lambda)^{1/2}\|_{HS}^2 d\lambda \right)^{1/2} \\ & \geq \sqrt{n} \left(\frac{\pi}{2} \right)^{(n+1)/2}. \end{aligned} \quad (13)$$

where $\hat{f}(\lambda)$ is the Fourier transform of f and $H(\lambda)$ are certain Hermite operators to be defined later.

All the three inequalities are easy to prove. A very simple proof of the inequality (1) can be found in Folland [1]. It so happens that we can simply adapt the same proof to all the three situations which we are interested in.

2. Uncertainty inequalities for Hermite and Laguerre expansions

The n dimensional Hermite functions are denoted by $\Phi_\alpha(x)$. They are eigen functions of the Hermite operator $H = (-\Delta + |x|^2)$.

$$(-\Delta + |x|^2)\Phi_\alpha(x) = (2|\alpha| + n)\Phi_\alpha(x). \quad (14)$$

Let P_N denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = N\}$. Then we can write

$$H^{1/2}f = \sum (2N + n)^{1/2} P_N f. \quad (15)$$

We also need to consider the operators A_j, A_j^+ which are defined by

$$A_j = \frac{\partial}{\partial \xi_j} + \xi_j, \quad A_j^+ = -\frac{\partial}{\partial \xi_j} + \xi_j. \quad (16)$$

These operators act on the Hermite functions in the following way:

$$\begin{aligned} A_j \Phi_\alpha &= (2\alpha_j)^{1/2} \Phi_{\alpha - e_j} \\ A_j^+ \Phi_\alpha &= (2\alpha_j + 1)^{1/2} \Phi_{\alpha + e_j} \end{aligned} \quad (17)$$

where $e_j = (0, 0, \dots, 1, \dots, 0)$ with 1 in the j th place. Then it is clear that the operators $H^{-1/2}A_j$ and $H^{-1/2}A_j^+$ are bounded on $L^2(\mathbb{R}^n)$.

Let $a \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$ be such that $\int |f(x)|^2 dx = 1$. Then we will prove the following theorem.

Theorem 2.1.

$$\left(\int |x - a|^2 |f(x)|^2 dx \right) \left(\sum_{N=0}^{\infty} (2N + n) \|P_N f\|_2^2 \right) \geq \frac{n}{4}.$$

Proof. The proof of this theorem is elementary. We start with the identity

$$\frac{\partial}{\partial x_j} \{(x_j - a_j)f\} = (x_j - a_j) \frac{\partial}{\partial x_j} f + f.$$

Then we have

$$\int f(x) \bar{f}(x) dx = \int \frac{\partial}{\partial x_j} \{(x_j - a_j)f\} \bar{f}(x) dx - \int (x_j - a_j) \bar{f}(x) \frac{\partial f}{\partial x_j} dx.$$

$$\int |f(x)|^2 dx = - \int (x_j - a_j) f(x) \frac{\partial \bar{f}}{\partial x_j} dx - \int (x_j - a_j) \bar{f}(x) \frac{\partial f}{\partial x_j} dx.$$

Now we write

$$(x_j - a_j) f \frac{\partial \bar{f}}{\partial x_j} = (x_j - a_j) f A_j \bar{f} - (x_j - a_j) x_j |f|^2$$

$$(x_j - a_j) \bar{f} \frac{\partial f}{\partial x_j} = - (x_j - a_j) \bar{f} A_j^+ f + (x_j - a_j) x_j |f|^2.$$

Using this it follows that

$$\int |f(x)|^2 dx = - \int (x_j - a_j) f A_j \bar{f} + \int (x_j - a_j) \bar{f} A_j^+ f.$$

Applying Cauchy-Schwarz inequality we get

$$\int |f(x)|^2 dx \leq \left(\int |x_j - a_j|^2 |f|^2 dx \right)^{1/2} (\|A_j f\|_2 + \|A_j^+ f\|_2).$$

Since $A_j H^{-1/2}$ and $A_j^+ H^{-1/2}$ are bounded on L^2 we get

$$\int |f(x)|^2 dx \leq 2 \left(\int |x_j - a_j|^2 |f(x)|^2 dx \right)^{1/2} \|H^{1/2} f\|_2.$$

Squaring and summing over j gives

$$\left(\int |x - a|^2 |f(x)|^2 dx \right) \left(\sum (2N + n) \|P_N f\|_2^2 \right) \geq \frac{n}{4}.$$

Let us now take up the case of the operator L . The operator L can be written as

$$L = -\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j)$$

where Z_j and $\bar{Z}_j$ are the vector fields defined by

$$Z_j = \partial_j + \frac{1}{4} \bar{z}_j, \quad \bar{Z}_j = \bar{\partial}_j - \frac{1}{4} z_j$$

where

$$\partial_j = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \bar{\partial}_j = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

We also have to recall some properties of the Weyl transform. For the Weyl transform we refer to Mauceri [3] and the references thereof.

The Weyl transform which we denote by W takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. There is a Plancherel formula for the Weyl transform:

$$\|f\|_2^2 = (2\pi)^{-n} \|W(f)\|_{\text{HS}}^2.$$

$$W(\bar{Z}_j f) = iW(f)A_j,$$

so that $W(L^{-1/2}Z_j f) = W(f)A_j^+ H^{-1/2}$ and $W(L^{-1/2}\bar{Z}_j f) = W(f)A_j H^{-1/2}$. The operators $Z_j L^{-1/2}$ and $\bar{Z}_j L^{-1/2}$ are easily seen to be bounded on $L^2(\mathbb{C}^n)$.

Let $a \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n)$ be normalized. Then we can prove the following inequality.

Theorem 2.2

$$\left(\int |z - a|^2 |f(z)|^2 dz \right) \left(\int |L^{1/2} f(z)|^2 dz \right) \geq \frac{n}{4}$$

Proof. The proof of this is very similar to the proof of the previous theorem. We start with the equation.

$$\partial_j \{ (z_j - a_j) f \} = f + (z_j - a_j) \partial_j f.$$

Proceeding as before we obtain

$$\int |f(z)|^2 dz = - \int (z_j - a_j) f (\bar{\partial}_j f) dz - \int (z_j - a_j) \bar{f} (\partial_j f) dz,$$

where we have used the fact that $(\partial_j \bar{f}) = \overline{(\bar{\partial}_j f)}$. Since $\partial_j = Z_j - \frac{1}{4} \bar{Z}_j$, $\bar{\partial}_j = \bar{Z}_j + \frac{1}{4} Z_j$ an easy calculation gives us

$$\int |f(z)|^2 dz = - \int (z_j - a_j) f (\bar{Z}_j f) dz - \int (z_j - a_j) \bar{f} (Z_j f) dz.$$

Applying Cauchy-Schwarz inequality and recalling that the operators $Z_j L^{-1/2}$ and $\bar{Z}_j L^{-1/2}$ are bounded on $L^2(\mathbb{C}^n)$ we get

$$\int |f(z)|^2 dz \leq 2 \left(\int |z_j - a_j|^2 |f(z)|^2 dz \right)^{1/2} \left(\int |L^{1/2} f(z)|^2 dz \right)^{1/2}.$$

Summing over j proves the theorem.

To deduce the uncertainty inequality for the Laguerre expansions we proceed as follows. When f is a radial function the Weyl transform reduces to the Laguerre transform:

$$W(f) = \sum_{N=0}^{\infty} R_N(f) P_N$$

where $R_N(f)$ are given by

$$R_N(f) = \frac{N!}{(N+n-1)!} \int_0^{\infty} f(r) L_N^{n-1}(\frac{1}{2}r^2) \exp(-\frac{1}{4}r^2) r^{2n-1} dr.$$

Since $W(L^{1/2} f) = W(f) H^{1/2}$ we have

$$W(L^{1/2} f) = \sum_{N=0}^{\infty} (2N+n)^{1/2} R_N(f) P_N$$

$$\|W(L^{1/2}f)\|_{\text{HS}}^2 = \sum_{N=0}^{\infty} (2N+n) \frac{(N+n-1)!}{N!} |R_N(f)|^2.$$

In view of the Plancherel formula we get

$$\|L^{1/2}f\|_2^2 = (2\pi)^{-n} \sum (2N+n) \frac{(N+n-1)!}{N!} |R_N(f)|^2.$$

Thus we have the following result.

Theorem 2.3. *For f in $L^2(\mathbb{R}_+, r^{2n-1} dr)$ with norm 1 we have the inequality*

$$\left(\int_0^\infty r^2 |f(r)|^2 r^{2n-1} dr \right) \left(\sum (2N+n) \frac{(N+n-1)!}{N!} |R_N(f)|^2 \right) \geq (2\pi)^n \frac{n}{4}.$$

If we take $n=1$ and define $f(r) = g(\sqrt{r})$ then we have

$$R_N(g) = \int_0^\infty f(r) L_N\left(\frac{1}{2}r\right) \exp\left(-\frac{1}{4}r\right) dr = \hat{f}(N)$$

and what we have is the following uncertainty inequality for the Laguerre expansions (of type 0).

$$\left(\int_0^\infty r^2 |f(r)|^2 dr \right) \left(\sum (2k+1) |\hat{f}(k)|^2 \right) \geq \frac{\pi}{2}.$$

3. Uncertainty inequality for the Heisenberg group

On the n -dimensional Heisenberg group H^n we consider the following left invariant vector fields

$$Z_j = \partial_j + i\bar{z}_j \partial_t, \quad \bar{Z}_j = \bar{\partial}_j - iz_j \partial_t, \quad T = \partial_t. \quad (18)$$

They generate the Lie algebra of H^n and the sublaplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = -\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \quad (19)$$

All the infinite dimensional irreducible unitary representations of H^n are parametrized by $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and they all can be realized on $L^2(\mathbb{R}^n)$. For all the facts about the representations π_λ of H^n and for other results which we use on H^n we refer to Geller [2] (see also Mauceri [4]).

The Fourier transform of a function f on H^n which we denote by $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$ for each $\lambda \neq 0$. The Fourier transform satisfies the following Plancherel's theorem:

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda)\|_{\text{HS}}^2 d\lambda. \quad (20)$$

Each representation π_λ determines a Lie algebra representation $d\pi_\lambda$. The operators $d\pi_\lambda(Z_j)$ and $d\pi_\lambda(\bar{Z}_j)$ are closable operators. Let $W_j(\lambda)$ and $\bar{W}_j(\lambda)$ denote their closures. Then they have a very simple description as follows let Φ_α^λ be the scaled Hermite functions defined by

$$\Phi_\alpha^\lambda(x) = (2|\lambda|^{1/2})^{n/2} \Phi_\alpha(2|\lambda|^{1/2}x). \quad (21)$$

Then Geller [2] has proved that for each $\lambda > 0$

$$W_j(\lambda)\Phi_\alpha^\lambda(x) = (2|\lambda|(\alpha_j + 1))^{1/2} \Phi_{\alpha + e_j}^\lambda(x) \quad (22)$$

$$\bar{W}_j(\lambda)\Phi_\alpha^\lambda(x) = (2|\lambda|\alpha_j)^{1/2} \Phi_{\alpha - e_j}^\lambda(x). \quad (23)$$

For $\lambda < 0$, $W_j(\lambda) = \bar{W}_j(-\lambda)$, $\bar{W}_j(\lambda) = W_j(-\lambda)$. Let $P_N(\lambda)$ denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda: |\alpha| = N\}$ and let $H(\lambda) = \sum_{N=0}^{\infty} (2N+n)|\lambda| P_N(\lambda)$. Then it is clear that the operators $H(\lambda)^{-1/2}W_j(\lambda)$ and $H(\lambda)^{-1/2}\bar{W}_j(\lambda)$ are bounded on $L^2(\mathbb{R}^n)$ for each $\lambda \neq 0$. We also have the identities

$$\begin{aligned} (Z_j f)^\wedge(\lambda) &= i\hat{f}(\lambda)W_j(\lambda) \\ (\bar{Z}_j f)^\wedge(\lambda) &= i\hat{f}(\lambda)\bar{W}_j(\lambda) \\ (Tf)^\wedge(\lambda) &= i\lambda\hat{f}(\lambda). \end{aligned} \quad (24)$$

Finally, when f is a zonal function i.e. a function of the form $f(|z|, t)$ then the Fourier transform $\hat{f}(\lambda)$ is explicitly given by the formula

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda) \quad (25)$$

where $R_N(\lambda, f)$ are defined by

$$R_N(\lambda, f) = \frac{N!}{(N+n-1)!} \int_0^\infty \tilde{f}(r, \lambda) L_N^{n-1}(2|\lambda|r^2) \exp(-(|\lambda|r^2)r^{2n-1}) dr. \quad (26)$$

In the above $\tilde{f}(r, \lambda)$ is the Fourier transform of f in the t variable and $L_N^{n-1}(r)$ are the Laguerre polynomials of type $(n-1)$.

Now for normalized L^2 functions f on H^n we can prove the following uncertainty inequality

$$\left(\iint t^2 |f(z, t)|^2 dz dt \right)^{1/2} \left(\int |\lambda|^n \|\lambda \hat{f}(\lambda)\|_{HS}^2 d\lambda \right)^{1/2} \geq \left(\frac{\pi}{2} \right)^{(n+1)/2} \quad (27)$$

The proof of this is elementary. First one proves that

$$\|f\|_2^2 \leq 2 \|tf\|_2 \|\partial_t f\|_2 \quad (28)$$

and then uses the Plancherel formula

$$\|\partial_t f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\lambda \hat{f}(\lambda)\|_{HS}^2 d\lambda \quad (29)$$

where we have used the relation $(\partial_t f)^\wedge(\lambda) = i\lambda\hat{f}(\lambda)$.

is true. In [5] Price and Sitarum have proved that if both the sets $\{t: f(z, t) \neq 0\}$ and $\{\lambda: f(\lambda) \neq 0\}$ have finite measure then $f \equiv 0$. But things are quite different when we consider f as a function of z .

Let f be zonal function viz. a function of the form $f(z, t) = \varphi(t)g(z)$ where g is radial. Then the Fourier transform of f is given by

$$\hat{f}(\lambda) = \tilde{\varphi}(\lambda) \sum_{N=0}^{\infty} R_N(\lambda, g) P_N(\lambda) \quad (30)$$

where

$$R_N(\lambda, g) = \frac{N!}{(N+n-1)!} \int_0^{\infty} g(r) L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) r^{2n-1} dr.$$

If g is compactly supported then by choosing $\tilde{\varphi}$ to have compact support we can make $\hat{f}(\lambda)$ also to have compact support. Nevertheless the following uncertainty inequality is valid.

Theorem 3.1. For a normalized L^2 function f on H^n one has

$$\begin{aligned} & \left(\iint |z - a|^2 |f(z, t)|^2 dz dt \right)^{1/2} \left(\int |\lambda|^n \|\hat{f}(\lambda)\| (H(\lambda))^{1/2} \|_{\text{HS}}^2 d\lambda \right)^{1/2} \\ & \geq \sqrt{n} \left(\frac{\pi}{2} \right)^{(n+1)/2}. \end{aligned}$$

Proof. Let $\tilde{f}(z, \lambda)$ denote the Fourier transform of f in the t variable. Then we have

$$\int |\tilde{f}(z, \lambda)|^2 dz = \int \partial_j((z_j - a_j)\tilde{f}) \bar{\tilde{f}} dz - \int (z_j - a_j)\tilde{f} \bar{\partial}_j \tilde{f} dz.$$

Integrating by parts in the first integral we get

$$\int |\tilde{f}(z, \lambda)|^2 dz = - \int (z_j - a_j)\tilde{f}(\bar{\partial}_j \tilde{f}) dz - \int (z_j - a_j)\tilde{f}(\partial_j \tilde{f}) dz.$$

Define $Z_j(\lambda)$ and $\bar{Z}_j(\lambda)$ by the equations

$$Z_j(\lambda) = \partial_j - \lambda \bar{z}_j, \quad \bar{Z}_j(\lambda) = \bar{\partial}_j + \lambda z_j. \quad (31)$$

Then a simple calculation shows that

$$\int |\tilde{f}(z, \lambda)|^2 dz = - \int (z_j - a_j)\tilde{f}(\bar{Z}_j(\lambda)\tilde{f}) dz - \int (z_j - a_j)\tilde{f}(Z_j(\lambda)\tilde{f}) dz.$$

But

$$Z_j(\lambda)\tilde{f}(z, \lambda) = (Z_j f)^{\sim}(z, \lambda)$$

$$\bar{Z}_j(\lambda)\tilde{f}(z, \lambda) = (\bar{Z}_j f)^{\sim}(z, \lambda)$$

and therefore,

$$\int |\tilde{f}(z, \lambda)|^2 dz = - \int (z_j - a_j)\tilde{f}(Z_j f)^{\sim} dz - \int (z_j - a_j)\tilde{f}(\bar{Z}_j f)^{\sim} dz.$$

Integrating with respect to λ , applying Cauchy-Schwarz and Plancherel theorem for the Fourier transform in the t variable we obtain

$$\|f\|_2^2 \leq \|(z_j - a_j)f\|_2 \{ \|Z_j f\|_2 + \|\bar{Z}_j f\|_2 \}. \quad (32)$$

In view of the Plancherel formula for the Heisenberg group

$$\|Z_j f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda) W_j(\lambda)\|_{\text{HS}}^2 d\lambda.$$

Using the fact that $H(\lambda)^{-1/2} W_j(\lambda)$ is bounded on $L^2(\mathbb{R}^n)$ we get

$$\|Z_j f\|_2 \leq \frac{2^{(n-1)/2}}{\pi^{(n+1)/2}} \left(\int |\lambda|^n \|\hat{f}(\lambda) (H(\lambda))^{1/2}\|_{\text{HS}}^2 d\lambda \right)^{1/2}. \quad (33)$$

Similarly

$$\|\bar{Z}_j f\|_2 \leq \frac{2^{(n-1)/2}}{\pi^{(n+1)/2}} \left(\int |\lambda|^n \|\hat{f}(\lambda) H(\lambda)^{1/2}\|_{\text{HS}}^2 d\lambda \right)^{1/2}. \quad (34)$$

Hence

$$\|f\|_2^2 \leq \left(\frac{2}{\pi} \right)^{(n+1)/2} \|(z_j - a_j)f\|_2 \left(\int |\lambda|^n \|\hat{f}(\lambda) H(\lambda)^{1/2}\|_{\text{HS}}^2 d\lambda \right)^{1/2}. \quad (35)$$

Summing over j proves the theorem.

If we use the fact that $(\mathcal{L}^{1/2} f)^\wedge(\lambda) = \hat{f}(\lambda) H(\lambda)^{1/2}$ then we get the inequality

$$\|z - a\|_2 \|f\|_2 \|\mathcal{L}^{1/2} f\|_2 \geq \sqrt{n} \left(\frac{\pi}{2} \right)^{(n+1)/2} \quad (36)$$

as advertised.

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Riesz means for the sublaplacian on the Heisenberg group

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Abstract. The uniform boundedness of the Riesz means for the sublaplacian on the Heisenberg group H^n is considered. It is proved that S_R^α are uniformly bounded on $L^p(H^n)$ for $1 \leq p \leq 2$ provided $\alpha > \alpha(p) = (2n+1)[(1/p) - (1/2)]$.

Keywords. Fourier transform; representations; sublaplacian; Riesz means; projections.

1. Introduction

Consider the Heisenberg group H^n which is a Lie group whose underlying manifold is $\mathbb{R} \times \mathbb{C}^n$ with coordinates (t, z) . Its Lie algebra $\mathfrak{h}^n$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j T, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j T. \quad (1)$$

Let $L = -\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j)$ be the sublaplacian on H^n . Then L is a formally nonnegative hypoelliptic differential operator which has a unique self adjoint extension to $L^2(H^n)$. Let $E(s)$ denote the spectral resolution of this extension which we normalize so as to become left continuous. $E(s)$ then becomes a right convolution operator with a C^∞ kernel.

In [5] Mauceri has studied the boundedness properties of the Riesz means S_R^α . These are defined for $\text{Re } \alpha > 0$ by the equation

$$S_R^\alpha f = \int_0^R \left(1 - \frac{s}{R}\right)^\alpha dE(s)f. \quad (2)$$

Let $\alpha(p) = (2n+1)[(1/p) - (1/2)]$ for $1 \leq p \leq 2$. Then Mauceri has proved, among other things, the following theorem.

Theorem. Assume that $1 \leq p \leq 2$ and $\alpha > \alpha(p)$. Then the uniform estimates

$$\|S_R^\alpha f\|_{L^p(H^n)} \leq C \|f\|_{L^p(H^n)} \quad (3)$$

hold for $f \in L^p(H^n)$.

Mauceri proved this theorem by getting L^p norm estimates for the kernel of the Riesz means. Here we propose to give a different proof of the above theorem using a pointwise estimate for the kernel.

Our proof follows an idea of Fefferman-Stein. In [1], in connection with a similar problem for the Laplacian on $\mathbb{R}^n$, it was shown that $L^p - L^2$ restriction properties of the Fourier transform imply the boundedness of the Riesz means. Later in [6] Chris Sogge has adapted the same techniques to prove the uniform boundedness of the Riesz means associated with self adjoint elliptic operators on a compact manifold. There it was proved that if we have good pointwise estimates of the kernel of S_R^α for large α and proper $L^p - L^2$ bounds for the projection operators associated with the Riesz means under consideration then we can get uniform L^p estimates for S_R^α itself. The same idea was used by this author in [9] to prove certain multiplier theorem for the Weyl transform.

Thus the plan of the paper is as follows. After collecting all preliminary materials in §2, we will get a good pointwise estimate for the kernel of S_R^α when α is an even integer in §3. Finally in §4 we take a partition of unity and write S_R^α as an infinite sum. For each piece there are certain 'projections' associated with it and for those projections we prove $L^p - L^2$ bounds. These bounds and the kernel estimate will then be used to prove that each piece is bounded on $L^p(H^n)$. The proof of the theorem will be completed by summing a geometric series.

2. Preliminaries

The main reference for this section is [2]. See also [5]. The $(2n+1)$ dimensional Heisenberg group is the Lie group whose underlying manifold is $\mathbb{R} \times C^n$. The group structure is given by

$$(s, z) \cdot (t, \zeta) = (s + t + 2 \operatorname{Im} z \cdot \bar{\zeta}, z + \zeta).$$

The Haar measure on H^n is simply the Lebesgue measure $ds dz d\bar{z}$ on $\mathbb{R} \times C^n$. If $w = (s, z)$ the homogeneous norm $|w|$ of w is defined by $|w|^4 = s^2 + |z|^4$. The homogeneous dimension of H^n is $Q = 2n + 2$.

Next we recall the definition of the Fourier transform on the Heisenberg group. The infinite dimensional representations of H^n are parametrized by $\mathbb{R} \setminus \{0\}$. If $\lambda \neq 0$ all the representations π_λ can be realized on the same Hilbert space, namely, $L^2(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n)$, $\pi_\lambda(s, z)$ is the operator defined by

$$\pi_\lambda(s, z)f(\xi) = \exp(i\lambda s) \exp(i2\lambda(2\xi - x) \cdot y) f(\xi - x) \quad (4)$$

where $z = x + iy \in C^n$ and $\xi \in \mathbb{R}^n$. The Fourier transform $\hat{f}$ of a function $f \in L^1(H^n)$ is then the operator valued function

$$\hat{f}(\lambda) = \int_{H^n} f(w) \pi_\lambda(w) dw. \quad (5)$$

We have the following Plancherel and inversion formulas:

$$\|f\|_2^2 = c_n \int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{\text{HS}}^2 |\lambda|^n d\lambda \quad (6)$$

where tr is the canonical semifinite trace and $\|\cdot\|_{\text{HS}}$ is the Hilbert-Schmidt norm.

For a bounded operator S on $L^2(\mathbb{R}^n)$ let us define $|S|_p$ by

$$|S|_p = (\text{tr}(S^*S)^{p/2})^{1/p}, \quad p < \infty$$

$$|S|_\infty = \text{operator norm of } S.$$

Let $\mathcal{L}_p$ denote the Banach space of weakly measurable operator valued functions $F: \mathbb{R} \setminus \{0\} \rightarrow B(L^2(\mathbb{R}^n))$ such that

$$\|F\|_{\mathcal{L}_p} = \left\{ C_n \int |F(\lambda)|_p^p |\lambda|^n d\lambda \right\}^{1/p} < \infty, \quad \text{for } p < \infty$$

$$\|F\|_{\mathcal{L}_\infty} = \text{ess-sup}_{\lambda \neq 0} |F(\lambda)|_\infty < \infty.$$

The Plancherel formula allows us to extend the Fourier transform as an isometry from $L^2(H^n)$ onto $\mathcal{L}_2$. More generally the following version of the Hausdorff-Young inequality holds (see [3]). Let $\mathcal{F}$ denote the inverse Fourier transform defined on $\mathcal{L}_1$ by

$$\mathcal{F}F(w) = \int \text{tr}(\pi_\lambda(w)^* F(\lambda)) |\lambda|^n d\lambda \quad (8)$$

and then extend $\mathcal{F}$ to $\mathcal{L}_p$ by interpolation for $1 < p < 2$. Then $\mathcal{F}$ maps $\mathcal{L}_p$, $1 \leq p \leq 2$ into $L^q(H^n)$, $(1/p) + (1/q) = 1$ and we have

$$\|\mathcal{F}F\|_q \leq C \|F\|_{\mathcal{L}_p}. \quad (9)$$

For each $\lambda \neq 0$ we can select an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $\Phi_\alpha^\lambda(x) = (2|\lambda|^{1/2})^{n/2} \Phi(2|\lambda|^{1/2}x)$ where Φ_α are the n -dimensional Hermite functions. Let $P_N(\lambda)$ denote the projection onto the subspace spanned by $\{\Phi_\alpha^\lambda: |\alpha| = N\}$. For a function $f(s, z) = f(s, |z|)$ which is called a zonal function, the Fourier transform reduces to the Laguerre transform and is given by

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda) \quad (10)$$

where $R_N(\lambda, f)$ are defined by

$$R_N(\lambda, f) = C_n \frac{N!}{(N+n-1)!} \int_0^\infty \tilde{f}(\lambda, r) L_N^{n-1}(2|\lambda|r^2) \\ \times \exp(-|\lambda|r^2) r^{2n-1} dr. \quad (11)$$

Here $\tilde{f}(\lambda, r)$ stands for the Euclidean Fourier transform and L_N^{n-1} are the Laguerre polynomials of type $n-1$. The inversion formula becomes

$$f(s, r) = \frac{2^{n-1}}{\pi^{n+1}} \int \exp(-i\lambda s) \left(\sum_{N=0}^{\infty} R_N(\lambda, f) L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) \right) |\lambda|^n d\lambda.$$

The Plancherel and the Hausdorff-Young theorem will then read as follows:

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int \sum_{N=0}^{\infty} |R_N(\lambda, f)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n d\lambda \quad (13)$$

$$\|f\|_q \leq C \left(\int \sum_{N=0}^{\infty} |R_N(\lambda, f)|^p \frac{(N+n-1)!}{N!} |\lambda|^n d\lambda \right)^{1/p}. \quad (14)$$

The Hausdorff-Young inequality in the above form will be used to prove the pointwise estimates for the kernel $S_R^\alpha(w)$ of the Riesz means.

Finally the following facts are also useful in the sequel. Each representation π_λ determines a Lie algebra representation $d\pi_\lambda$. The operators $d\pi_\lambda(Z_j)$ and $d\pi_\lambda(\bar{Z}_j)$ are closable operators. If $W_f(\lambda)$ and $\bar{W}_f(\lambda)$ denote the closures then we have

$$W_f(\lambda)\Phi_\alpha^\lambda = (2|\lambda|(\alpha_j + 1))^{1/2} \Phi_{\alpha+e_j}^\lambda,$$

$$\bar{W}_f(\lambda)\Phi_\alpha^\lambda = (2|\lambda|\alpha_j)^{1/2} \Phi_{\alpha-e_j}^\lambda$$

where e_j are the the coordinate vectors in $\mathbb{R}^n$. Let $H_j(\lambda) = -\frac{1}{2}(W_j(\lambda)\bar{W}_j(\lambda) + \bar{W}_j(\lambda)W_j(\lambda))$ and $H(\lambda) = \sum_{j=1}^n H_j(\lambda)$. Then $H(\lambda)$ is the closure of $d\pi_\lambda(L)$ and has the spectral resolution

$$H(\lambda) = \sum_{N=0}^{\infty} (2N+n)|\lambda| P_N(\lambda).$$

More generally one has

$$d\pi_\lambda(\phi(L)) = \sum_{N=0}^{\infty} \phi((2N+n)|\lambda|) P_N(\lambda) \quad (15)$$

for any reasonable function ϕ .

3. Pointwise estimates for the Riesz kernel

The aim of this section is to prove the following estimate for the Riesz kernel.

Theorem 3.1. *Assume that $\alpha > 2m$ where m is a positive integer. Then the kernel $S_R^\alpha(w)$ of the Riesz means satisfies the following estimate*

$$|S_R^\alpha(w)| \leq CR^{Q/2} (1 + R^{1/2}|w|)^{-2m}. \quad (16)$$

Recall that $S_R^\alpha f = \phi_R^\alpha(L)f$ where $\phi_R^\alpha(\lambda) = (1 - (\lambda/R))_+^\alpha$. Therefore, $S_R^\alpha f = S_R^\alpha * f$ where the Fourier transform of the kernel S_R^α is given by the equation

$$(S_R^\alpha)^\wedge(\lambda) = \sum_{N=0}^{\infty} \left(1 - \frac{(2N+n)|\lambda|}{R} \right)_+^\alpha P_N(\lambda). \quad (17)$$

If we apply the inversion formula (12) we get

$$\times \left\{ \sum_{N=0}^{\infty} \left(1 - \frac{(2N+n)|\lambda|}{R} \right)_+^{\alpha} L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) \right\} |\lambda|^n d\lambda \quad (18)$$

where we have set $w=(s, z)$, $r=|z|$. From the above expression it is clear that $S_R^{\alpha}(s, z) = R^{Q/2} S_1^{\alpha}(Rs, R^{1/2}z)$ and hence it is enough to estimate $S_1^{\alpha}(w)$.

Since $S_1^{\alpha}(w)$ is the inverse Fourier transform of the operator valued function $(1 - H(\lambda))_+^{\alpha}$, Hausdorff-Young inequality gives the estimate

$$\|S_1^{\alpha}(w)\|_{\infty} \leq C \|(1 - H(\lambda))_+^{\alpha}\|_{\mathcal{S}^1}.$$

As $(1 - H(\lambda))_+^{\alpha} = \sum_{N=0}^{\infty} (1 - (2N+n)|\lambda|)_+^{\alpha} P_N(\lambda)$, it is easy to see that $\|(1 - H(\lambda))_+^{\alpha}\|_{\mathcal{S}^1} \leq C$ and hence $\|S_1^{\alpha}(w)\|_{\infty} \leq C$. Thus it is enough to prove the estimate $|S_1^{2m}(w)| \leq c|w|^{-1-2m}$ when $|w| > 1$.

Let us set $f = S_1^{\alpha}$ so that $\hat{f}(\lambda)$ is given by $\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda)$. Here $R_N(\lambda, f) = (1 - (2N+n)|\lambda|)_+^{\alpha} = \psi((2N+n)|\lambda|)$, say. Our aim is to calculate $R_N(\lambda, (is - r^2)^m f)$ and then apply the Hausdorff-Young inequality getting

$$\|(is - r^2)^m f\|_{\infty} \leq C \sum_{N=0}^{\infty} \int |R_N(\lambda, (is - r^2)^m f)| \frac{(N+n-1)!}{N!} |\lambda|^n d\lambda. \quad (19)$$

If we can show that the right hand side of (19) is bounded then we are done.

Recall the definition of $R_N(\lambda, f)$. We have

$$R_N(\lambda, f) = c_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(\lambda, r) L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) r^{2n-1} dr.$$

Since $(isf)^{\sim}(\lambda, r) = (d/d\lambda) \tilde{f}(\lambda, r)$ we have the formula

$$\begin{aligned} R_N(\lambda, isf) &= \frac{d}{d\lambda} R_N(\lambda, f) - c_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(\lambda, r) \\ &\quad \times \frac{d}{d\lambda} \{L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2)\} r^{2n-1} dr. \end{aligned}$$

Now we make use of the following formula satisfied by the Laguerre polynomials (see [7]).

$$r \frac{d}{dr} L_N^{n-1}(r) = N L_N^{n-1}(r) - (N+n-1) L_{N-1}^{n-1}(r).$$

Using this formula, a simple calculation shows that for $\lambda \neq 0$ we have

$$R_N(\lambda, isf) = \frac{d}{d\lambda} R_N(\lambda, f) - N|\lambda|^{-1} \{R_N(\lambda, f) - R_{N-1}(\lambda, f)\} + R_N(\lambda, r^2 f).$$

So, if we let $\sigma = (2N+n)|\lambda|$ and $\psi_1(\sigma) = R_N(\lambda, (is - r^2)f)$ then we have proved the following formula

$$\psi_1(\sigma) = |\lambda|^{-1} \{\sigma \psi'(\sigma) - N\psi(\sigma) + N\psi(\sigma - 2|\lambda|)\}.$$

Let us now recall that $\psi(\sigma) = (1 - \sigma)_+^{\alpha}$ satisfies the following two properties: i) ψ is supported in $(2N+n)|\lambda| \leq 1$; ii) $\int |\psi((2N+n)|\lambda)| |\lambda|^n d\lambda \leq C(2N+n)^{-n-1}$.

From the above equation for $\psi_1(\sigma)$ it is clear that ψ_1 also satisfies the property (i). We claim that ψ_1 satisfies property (ii) also. To see this we have

$$\begin{aligned} \psi_1((2N+n)|\lambda|) &= \frac{1}{2}n|\lambda|^{-1} \frac{\partial \psi}{\partial N} + 2N|\lambda|^{-1} \\ &\times \left\{ \frac{\partial \psi}{\partial N} - \psi((2N+n)|\lambda|) - \psi((2(N-1)+n)|\lambda|) \right\}. \end{aligned}$$

Using Taylor expansion we can write

$$\begin{aligned} &\psi((2(N-1)+n)|\lambda|) - \psi((2N+n)|\lambda|) + \frac{\partial \psi}{\partial N} \\ &= 4|\lambda|^2 \int_{N-1}^N (t+1-N) \psi''((2t+n)|\lambda|) dt. \end{aligned}$$

Since $(2N+n)|\lambda| \leq 1$ we get

$$\begin{aligned} &\int 2N|\lambda|^{-1} \left| \psi((2(N-1)+n)|\lambda|) - \psi((2N+n)|\lambda|) + \frac{\partial \psi}{\partial N} \right| |\lambda|^n d\lambda \\ &\leq C \int_{N-1}^N dt \int (1-(2t+n)|\lambda|)_+^{\alpha-2} |\lambda|^n d\lambda \leq C(2N+n)^{-n-1}. \end{aligned}$$

We also have

$$\int |\lambda|^{-1} \left| \frac{\partial \psi}{\partial N} \right| |\lambda|^n d\lambda \leq C(2N+n)^{-n-1}.$$

Hence we have proved that

$$\int |\psi_1((2N+n)|\lambda|)| |\lambda|^n d\lambda \leq C(2N+n)^{-n-1}.$$

Now an iteration of the process shows that $R_N(\lambda, (is-r^2)^kf) = \psi_k$ satisfies the inequality

$$\int |\psi_k((2N+n)|\lambda|)| |\lambda|^n d\lambda \leq C_k(2N+n)^{-n-1}$$

provided $(1-\lambda)_+^{\alpha-2k}$ is integrable. Hence when $k=m$ we get

$$\int |R_N(\lambda(is-r^2)^mf)| |\lambda|^n d\lambda \leq C(2N+n)^{-n-1}$$

and consequently

$$\sum_{N=0}^{\infty} \int |R_N(\lambda(is-r^2)^mf)| \frac{(N+n-1)!}{N!} |\lambda|^n d\lambda \leq C \sum_{N=0}^{\infty} (2N+n)^{-2} \leq C.$$

$$\|(is - r^2)^m f\|_\infty \leq C$$

and hence this completes the proof of Theorem 3.1.

4. Boundedness of the Riesz means

In this section we will prove our main theorem. By rescaling we can assume that $R = 1$. So we are considering the operator $T = (1 - L)_+^\alpha$. We are going to prove that when $\alpha > \alpha(p)$

$$\|Tf\|_p \leq C \|f\|_p, \quad f \in L^p(H^n). \quad (20)$$

To that end we take a partition of unity $\sum_{j=-\infty}^{\infty} \phi(2^j s) = 1$ where $\phi \in C_0^\infty(\frac{1}{2}, 2)$. Let us write $\phi_j^\alpha(s) = (1 - s)^\alpha \phi(2^j(1 - s))$ and define

$$T_j = \int_0^1 \phi_j^\alpha(s) dE(s). \quad (21)$$

Then $T = \sum_{j=0}^{\infty} T_j$ and (20) will follow once we prove the following proposition:

PROPOSITION 4.1

Assume that $\alpha > \alpha(p)$ and $f \in L^p(H^n)$. Then $\exists \varepsilon > 0$ such that

$$\|T_j f\|_p \leq C 2^{-j\varepsilon} \|f\|_p. \quad (22)$$

We claim that Proposition 4.1 is a consequence of the following estimates:

$$\|T_j f\|_2 \leq C 2^{-\alpha j} 2^{-j((1/p) - (1/2))} \|f\|_p. \quad (23)$$

This estimate will be proved in the next proposition. To complete the proof of (22) we follow Sogge [6] and therefore will be sketchy in the proof.

We first show that whenever B is a ball of radius 2^j then

$$\|T_j f\|_{L^p(B)} \leq C 2^{-j(\alpha - \alpha(p))} \|f\|_p. \quad (24)$$

This is immediate since

$$\|T_j f\|_{L^p(B)} \leq C 2^{jQ((1/p) - (1/2))} \|T_j f\|_2$$

in view of the estimate (23). We next show that the kernel of T_j has ‘thickness’ 2^j . More precisely it is enough to show that for each $\nu > 0 \exists$ an $\varepsilon > 0$ for which one has the uniform estimates

$$\int_{|w| > 2^{j(1+\nu)}} |S_j^\alpha(w)| dw \leq C 2^{-j\varepsilon}$$

where S_j^α is the kernel of T_j . To prove this estimate we make use of the kernel estimate

$$|S_R^{2m+1}(w)| \leq CR^{Q/2}(1 + R^{1/2}|w|)^{-2m}. \quad (25)$$

Recall that $T_j = \int_0^1 \phi_j(s) dE(s)$, $\phi_j(s) = (1-s)^j \phi(2^j(1-s))$.

Proceeding as in [6] one obtains the following equation for the kernel S_j^α

$$S_j^\alpha(w) = c_m \int t^{2m+1} S_t^{2m+1}(w) \partial_t^{2m+2} \phi_j^\alpha(t) dt.$$

Since ϕ_j^α is supported in $2^{-j-1} \leq (1-t) \leq 2^{-j+1}$ we have the bound

$$|\partial_t^{2m+2} \phi_j^\alpha(t)| \leq C 2^{j(2m+2)}.$$

If we use the estimate (25) what we get is

$$|S_j^\alpha(w)| \leq C 2^{j(2m+2)} |w|^{-2m}.$$

From the estimate it follows that

$$\begin{aligned} \int_{|w| > 2^{j(1+v)}} |S_j^\alpha(w)| dw &\leq 2^{j(2m+2)} \int_{2^{j(1+v)}}^\infty t^{-2m+Q-1} dt \\ &\leq C 2^{j(2m+2)} 2^{j(1+v)(-2m+Q)}. \end{aligned}$$

Choosing m large so that $(2m+2)v > (Q+1)(v+1)$ we can arrange it so that

$$\int_{|w| > 2^{j(1+v)}} |S_j^\alpha(w)| dw \leq C 2^{-j\epsilon}. \quad (26)$$

Now using (24) and (26) one can prove (22). A detailed proof of this fact can be found in [8]. We omit the details here and proceed to the proof of the following:

PROPOSITION 4.2.

Assume that $1 \leq p \leq 2$ and $f \in L^p(H^n)$. Then we have

$$\|T_j f\|_2 \leq C 2^{-\alpha j} 2^{-j((1/p)-(1/2))} \|f\|_p. \quad (27)$$

Since $\|T_j f\|_2 \leq C 2^{-\alpha j} \|f\|_2$ it is enough to prove the inequality when $p=1$. The proof of the inequality is simple when $p=1$. In view of the Plancherel theorem we have

$$\|T_j f\|_2^2 = \int \|(T_j f)^\wedge(\lambda)\|_{HS}^2 |\lambda|^n d\lambda.$$

Since $T_j = \varphi_j^\alpha(L)$ we have the equation

$$(T_j f)^\wedge(\lambda) = \sum_{N=0}^{\infty} \varphi_j^\alpha(v|\lambda|) P_N(\lambda) \hat{f}(\lambda)$$

where $v = (2N+n)$. Due to the orthogonality of the projections $P_N(\lambda)$ we have

If we define the function $\varphi_{Nj}(t, |z|) = \varphi_{Nj}(t, r)$ by the equation

$$\varphi_{Nj}(t, r) = \int \exp(-i\lambda t) \varphi_j^\alpha(v|\lambda|) L_N^{n-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) |\lambda|^n d\lambda$$

then it is easy to see that $\hat{\varphi}_{Nj}(\lambda) = \varphi_j^\alpha(v|\lambda|) P_N(\lambda)$ so that

$$\|T_j f\|_2^2 = \sum_{N=0}^{\infty} \|\varphi_{Nj} * f\|_2^2. \quad (28)$$

By applying Young's inequality we obtain

$$\|T_j f\|_2^2 \leq c \|f\|_1^2 \sum_{N=0}^{\infty} \|\varphi_{Nj}\|_2^2.$$

Now it is an easy matter to calculate $\|\varphi_{Nj}\|_2$. Recalling the definition of φ_{Nj} we have

$$\begin{aligned} \|\varphi_{Nj}\|_2^2 &= \iint_0^\infty |\varphi_j^\alpha(v|\lambda|)|^2 |\lambda|^{2n} (L_N^{n-1}(2|\lambda|r^2) \exp(-\lambda r^2))^2 r^{2n-1} dr d\lambda \\ &= \iint_0^\infty |\varphi_j^\alpha(v|\lambda|)|^2 |\lambda|^n (L_N^{n-1}(r^2))^2 \exp(-\lambda r^2) r^{2n-1} dr d\lambda. \end{aligned}$$

Since $\varphi_j^\alpha(s) = (1-s)^\alpha \varphi(2^j(1-s))$, on the support of φ_j^α we have $2^{-j-1} \leq (1-s) \leq 2^{-j+1}$ and hence $(1-v|\lambda|)^\alpha \leq c 2^{-j\alpha}$. We also have

$$\int_0^\infty (L_N^{n-1}(r^2))^2 \exp(-r^2) r^{2n-1} dr \leq c v^{n-1}$$

which is a basic property of the Laguerre functions. Therefore, we have

$$\begin{aligned} \|\varphi_{Nj}\|_2^2 &\leq c v^{n-1} 2^{-2j\alpha} \int |\varphi(2^j(1-v|\lambda|))|^2 |\lambda|^n d\lambda \\ &\leq c 2^{-2j\alpha} 2^{-j\nu-2}. \end{aligned} \quad (29)$$

Hence we have proved

$$\|T_j f\|_2^2 \leq c \|f\|_1^2 2^{-2\alpha j} 2^{-j} \sum_{N=0}^{\infty} (2N+n)^{-2}$$

or

$$\|T_j f\|_2 \leq c 2^{-\alpha j} 2^{-j/2} \|f\|_1. \quad (30)$$

This completes the proof of Proposition 4.2.

Note: After this paper was written the author came to know that D. Muller has obtained the same result by using almost similar methods in the paper 'On Riesz means of eigenfunction expansions for the Kohn-Laplacian' (preprint).

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Pairs of II_1 factors

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Abstract. A class of objects – that are best described as being actions of *group-like objects of von Neumann algebras* – is axiomatised and it is shown that there exists a bijective correspondence between isomorphism classes of such *covariant systems* and isomorphism classes of pairs of II_1 factors (M, N) satisfying $N \subset M$, $[M:N] < \infty$ and $M \cap N' = \mathbb{C}$.

Keywords. Irreducible subfactor; group-like object; covariant system.

1. Introduction

The aim of this paper is to associate with each pair (M, N) of II_1 factors satisfying $N \subset M$, $[M:N] < \infty$ and $M \cap N' = \mathbb{C}$, a group-like object $\mathcal{G}_{M/N}$ (or rather, an isomorphism class of such objects) in such a way that

- (i) $(M, N) \cong (\tilde{M}, \tilde{N})$ if and only if $\mathcal{G}_{M/N} \cong \mathcal{G}_{\tilde{M}/\tilde{N}}$; and
- (ii) from each such object $\mathcal{G}$ that is abstractly defined, to construct a pair $(N \times \mathcal{G}, N)$ of II_1 factors (with finite index and trivial relative commutant, as above) in such a way that (a) $\mathcal{G}_{N \times \mathcal{G}/N} \cong \mathcal{G}$, and (b) $(N \times \mathcal{G}_{M/N}, N) \cong (M, N)$.

The object $\mathcal{G}$ is quite different in character from the one discussed by Ocneanu in [1] and [3]; we refer to objects such as $\mathcal{G}$ simply as covariant systems. Our notion of a covariant system (as in Definition (4.1)) is defined by five axioms, three of which are similar to the group axioms, while the other two seem to correspond to the definition of a group action. Unfortunately, the author has been unable to decouple the notions of ‘group-like object’ and ‘actions of such objects’; it is not clear whether the failure to do so is a consequence of the author’s ineptness or of some inherent feature of this approach (which amounts to identifying axioms that yield the Hilbert algebra structure of the extension M purely in terms of the subfactor N and some morphisms associated with N).

The third section is devoted to ‘right bases for M over N ’, a slight generalization of the ‘orthogonal bases’ introduced in [5]; the reason for this generalization is that orthogonality gets destroyed under certain natural constructions involving products. This was most probably noticed by Ocneanu, since several results of this section occur with only slight and notational changes in [3]. This section is included here for completeness, and since there are few proofs in [3]. Thus, the first half (up till Proposition 6) of this section is a modified reproduction of parts of [5], while the second half ([Proposition 7, Proposition 11]) is essentially contained in [3].

The fourth section is the core of this paper, and is devoted to attaining the goals

descriptions of the C*-algebras corresponding to G/H and G/H , where H is a (not necessarily normal) subgroup of a finite (not necessarily abelian) group G .

Finally, the author would like to attempt to describe the nature of the relationship between this work and Ocneanu's work. The two approaches (to the same problem) are very similar up to a point, after which they diverge; the common part is that part which builds on Pimsner-Popa's discussion of bases, by starting with a basis for M over N and proceeding to construct bases over N for each member M_n of the tower of the basic construction. Thus, while it is true that the author has benefited greatly from numerous conversations with Ocneanu and that this work has been strongly influenced, in spirit, by Ocneanu's work, this paper relies more heavily, in content, on the paper [5] of Pimsner and Popa.

2. Preliminaries

We shall be concerned with the category $\mathcal{C} = \{(M, N) : M, N \text{ II}_1 \text{ factors, } N \subseteq M, [M:N] < \infty\}$ and its subcategory

$$\mathcal{C}_0 = \{(M, N) \in \mathcal{C} : M \cap N' = C\},$$

where, of course a morphism $\pi : (M, N) \rightarrow (\tilde{M}, \tilde{N})$ is a unital normal $*$ -homomorphism $\pi : M \rightarrow \tilde{M}$ such that $\pi(N) \subseteq \tilde{N}$. We begin by recalling some facts from [1], [5] and [6].

If $(M, N) \in \mathcal{C}$, there is a notion of an extension M_1 of M by N , which has the following features:

(P1) $(M_1, M) \in \mathcal{C}$ and $[M_1 : M] = [M : N]$

(P2) There exists a projection $e_0 \in M_1$ such that

(i) $e_0 x e_0 = E_N(x) e_0$, for all $x \in M$; (ii) $M_1 = \langle M, e_0 \rangle (= (M \cup \{e_0\})''$); (iii) $E_M(e_0) = \tau \cdot 1$. (Here and in the sequel, the symbol τ denotes $[M : N]^{-1}$, the symbol tr always stands for the unique normalized trace, and E_N denotes the unique tr -preserving conditional expectation of any II_1 factor onto a subfactor N .)

(P3) For each $x \in M_1$, there is a unique m in M such that $x e_0 = m e_0$; in view of (P2) (iii), we necessarily have $m = \tau^{-1} E_M(x e_0)$.

If $(M, N) \in \mathcal{C}$, we shall let $\{M_n\}_{n=-1}^\infty$ denote the tower obtained by iterating the basic construction; thus $M_{-1} = N$, $M_0 = M$ and M_{n+1} is the extension of M_n by M_{n-1} ; we shall let e_n be a projection in M_{n+1} that implements the conditional expectation of M_n onto M_{n-1} . The sequence $\{e_n\}_{n=0}^\infty$ of projections satisfies the following 'Jones relations':

(P4) (i) $e_m e_n = e_n e_m$ for $|m - n| > 1$; (ii) $e_n e_m e_n = \tau e_n$ if $|m - n| = 1$; (iii) $E_{M_n}(e_n) = \tau \cdot 1$

(P5) For $n \geq 1$, M_{2n+1} is the extension of M_n by M_{-1} , a choice of a projection in M_{2n+1} that implements the conditional expectation of M_n onto M_{-1} being given by $e_{(n)} = \tau^{-n(n+1)/2} (e_n e_{n-1} \cdots e_0) (e_{n+1} e_n \cdots e_1) \cdots (e_{2n} e_{2n-1} \cdots e_n)$.

We end this preliminary section by introducing some notation we shall employ. Symbols I, J, K , etc. will denote finite sets. We shall work with the algebra $\text{Mat}_{I \times J}(N)$ of matrices, with entries in N , and rows and columns indexed by I and J respectively. An element of $\text{Mat}_{I \times J}(N)$ will be denoted by $x = ((x_{ij}))$; when $I = J$, clearly $\text{Mat}_{I \times I}(N)$ is II_1 factor, the equation $\text{Tr } x = \sum_i \text{Tr } x_{ii}$ defining a non-normalized trace on $\text{Mat}_{I \times I}(N)$. Row-vectors (respectively, column vectors) will be denoted by $\text{Mat}_{\times I}(N)$ (respectively, $\text{Mat}_{I \times \times}(N)$).

In a natural fashion, we have $\text{Mat}_{I \times J}(N) \subseteq \text{Mat}_{I \times J}(M) \subseteq \text{Mat}_{I \times J}(M_1) \subseteq \dots$. If $a \in \bigcup_n M_n$ and $x \in \bigcup_n \text{Mat}_{I \times J}(M_n)$, we shall let axb denote the matrix defined by $axb = ((ax_i b_j))$. Also, if $n < m$, we shall let the same symbol E_{M_n} denote the map from $\text{Mat}_{I \times J}(M_n)$ into $\text{Mat}_{I \times J}(M_m)$ defined by applying E_{M_n} entrywise. It must be obvious that the statements (P2)(i), (P2)(iii) and (P3) have the following matricial analogues

(P2)(i) $_{\text{Mat}}$ $e_0 x e_0 = E_N(x) e_0$ for all $x \in \text{Mat}_{I \times J}(M)$;

(P2)(iii) $_{\text{Mat}}$ $\text{Tr } e_0 x = \tau \text{Tr } x$ for all $x \in \text{Mat}_{I \times J}(M)$;

(P3) $_{\text{Mat}}$ For each $x \in \text{Mat}_{I \times J}(M_1)$, there is a unique $m \in \text{Mat}_{I \times J}(M)$ such that $x e_0 = m e_0$ necessarily $m = \tau^{-1} E_M(x e_0)$.

3. Right bases

We shall assume henceforth that $(M, N) \in \mathcal{C}$ and that $\{M_n\}$ is the associated tower of the basic construction, and I will denote a finite set.

Lemma 1. The following conditions on a column-vector $\lambda = ((\lambda_i)) \in \text{Mat}_{I \times 1}(M)$ are equivalent:

(i) $\lambda^* e_0 \lambda = 1$; (ii) $E_N(\lambda \lambda^*) = Q_\lambda$ is a projection in $\text{Mat}_{I \times I}(N)$ satisfying $\text{Tr } Q_\lambda = [M:N]$

Proof. Fix $i_0 \in I$ and define $v \in \text{Mat}_{I \times I}(N)$ by $v_{ij} = \delta_{i_0 j} e_0 \lambda_{i_0}$, where of course δ denotes the Kronecker symbol. Then $(v^* v)_{ij} = \delta_{i_0 i} \delta_{i_0 j} \lambda^* e_0 \lambda$, while $vv^* = Q_{\lambda} e_0$. Hence the assumption (i) is equivalent to the assumption that v is a partial isometry with $\text{Tr } v^* v = 1$, while (ii) amounts to requiring that v is a partial isometry with $\text{Tr } vv^* = \tau \text{Tr } Q_\lambda = 1$ (cf. (P2)(iii) $_{\text{Mat}}$); thus, (i) is equivalent to (ii).

DEFINITION 2

A column vector $\lambda \in \text{Mat}_{I \times 1}(M)$ is called a right basis for M/N if λ satisfies either of the equivalent conditions of Lemma 1.

It should be remarked that this notion of right basis is a marginal generalization of the 'orthogonal basis' considered in [5]; a right basis λ (in the sense of Definition 2) is an orthogonal basis in the sense of [5] precisely when the associated projection Q_λ is diagonal. The reason for relaxing orthogonality will become evident later.

PROPOSITION 3. (Existence of right bases)

Let I be any finite set (with $|I|$ necessarily at least as large as $[M:N]$). and let Q be a projection in $\text{Mat}_{I \times I}(N)$ such that $\text{Tr } Q = [M:N]$. Then there exists a right basis $\lambda \in \text{Mat}_{I \times 1}(M)$ for M/N such that $E_N(\lambda \lambda^*) = Q$.

Proof. Define $P \in \text{Mat}_{I \times I}(N)$ by $P_{ij} = \delta_{i_0 j} \delta_{i_0 i}$ where $i_0 \in I$ is fixed. Then $\text{Tr } P = 1$, and also $\text{Tr } Q e_0 = 1$. Since $Q e_0$ and P are projections in the II_1 factor $\text{Mat}_{I \times I}(M_1)$ with equal trace, there exists a partial isometry $v \in \text{Mat}_{I \times I}(M_1)$ such that $v^* v = P$ and $vv^* = Q e_0$. Since $v^* v = P$, it follows that $v_{ij} = 0$ if $j \neq i_0$; so there exist $v_i \in M_1$ such that $v_{ij} = \delta_{i_0 j} v_i$; the assumption $vv^* = Q e_0$ implies that $v = e_0 v$ and hence $v_i = e_0 v_i$ for each i . Appeal now to (P3) to deduce the existence of $\lambda_i \in M$ such that $e_0 \lambda_i = e_0 v_i = v_i$. I

follows at once that $q_{ij}e_0 = (vv^*)_{ij} = e_0\lambda_i\lambda_j^*e_0 = E_N(\lambda_i\lambda_j^*)e_0$; it follows that $Q = E_N(\lambda\lambda^*)$, and that λ is necessarily a right basis for M/N . $\square$

DEFINITION 4

If λ is a right basis for M/N , define $\eta: M \rightarrow \text{Mat}_{I \times I}(N)$ by $\eta(x) = E_N(x\lambda^*)$; when it is necessary to indicate the dependence of η on the choice of λ , we shall write $\eta = \eta_\lambda$.

PROPOSITION 5

Let λ be as in Definition 2 and let $Q = E_N(\lambda\lambda^*)$.

- (i) If $x \in M$, then $x = \eta(x)\lambda$ and $\eta(x) = \eta(x)Q$.
- (ii) If $x \in M$ and if $x = \xi\lambda$ where $\xi \in \text{Mat}_{I \times I}(N)$ satisfies $\xi = \xi Q$, then, $\xi = \eta(x)$.

Proof. Start by noticing that if v is the partial isometry as in the proof of Lemma 1, then

$$\begin{aligned} e_0\lambda_i &= v_{ii_0} = (Qe_0v)_{ii_0} = \sum_j q_{ij}e_0\lambda_j \\ &= e_0(Q\lambda)_i \text{ and hence } \lambda = Q\lambda. \end{aligned} \quad (1)$$

- (i) $e_0x = e_0x \cdot \lambda^*e_0\lambda = e_0E_N(x\lambda^*)\lambda$ and so (P3) ensures that $x = \eta(x)\lambda$; also,

$$\eta(x)Q = E_N(x\lambda^*)Q = E_N(x\lambda^*Q) = E_N(x\lambda^*) \text{ by (1).}$$

- (ii) If $x = \xi\lambda$ where $\xi = \xi Q$, then

$$E_N(x\lambda^*) = E_N(\xi\lambda\lambda^*) = \xi E_N(\lambda\lambda^*) = \xi Q = \xi. \quad \square$$

PROPOSITION 6. ('Uniqueness' of right bases)

If $\lambda \in \text{Mat}_{I \times I}(M)$ and $\tilde{\lambda} \in \text{Mat}_{\tilde{I} \times \tilde{I}}(M)$ are right bases for M/N , then there exists a partial isometry $u \in \text{Mat}_{I \times \tilde{I}}(N)$ such that

- (i) $u^*u = Q_{\tilde{\lambda}}$, $uu^* = Q_\lambda$; (ii) $\lambda = u\tilde{\lambda}$, $\tilde{\lambda} = u^*\lambda$.

Proof. Put $u = E_n(\lambda\tilde{\lambda}^*)$ and apply the previous proposition to both λ and $\tilde{\lambda}$. Thus, for instance, since $\tilde{\lambda}$ is a right basis for M/N , it follows that

$$\lambda = E_N(\lambda\tilde{\lambda}^*)\tilde{\lambda} = u\tilde{\lambda}.$$

The other assertions are proved as painlessly. $\square$

For the rest of this section, assume that $(M, N) \in \mathcal{C}$ and that $\lambda \in \text{Mat}_{I \times I}(M)$ is a right basis for M/N .

PROPOSITION 7

- (i) $\tau^{-1/2}e_0\lambda \in \text{Mat}_{I \times I}(M_1)$ is a right basis for M_1/M ;
- (ii) If $\lambda^{(2)} \in \text{Mat}_{I^2 \times I^2}(M_1)$ is defined by $\lambda_i^{(2)} = \tau^{-1/2}\lambda_{i_1}e_0\lambda_{i_2}$ (for $i = (i_1, i_2) \in I^2$), then $\lambda^{(2)}$ is a right basis for M_1/N .
- (iii) More generally than (ii), if for $n = 1, 2, \dots$, $\lambda^{(n)} \in \text{Mat}_{I^n \times I^n}(M_{n-1})$ is defined by

$$\lambda_i^{(n)} = \tau^{-n(n-1)/4}\lambda_{i_1}e_0\lambda_{i_2}e_1e_0\lambda_{i_3}\cdots\lambda_{i_{n-1}}e_{n-2}e_{n-3}\cdots e_0\lambda_{i_n},$$

then $\lambda^{(n)}$ is a right basis for M_{n-1}/N .

Proof. (i) $(\tau^{-1/2}e_0\lambda)^*e_1(\tau^{-1/2}e_0\lambda) = \tau^{-1}\lambda^*e_0e_1e_0\lambda = \lambda^*e_0\lambda = 1$. (cf. (P4)(ii))

(iii) We have to prove that $\lambda^{(n)*}e_{(n-1)}\lambda^{(n)} = 1$ for all n , where $e_{(n)}$ is as in (P5). When $n = 1$, $\lambda^{(1)} = \lambda$ and $e_{(0)} = e_0$, and $\lambda^*e_0\lambda = 1$ by definition of λ .

Assume now that $\lambda^{(n)*}e_{(n-1)}\lambda^{(n)} = 1$, so we need to show that $\lambda^{(n+1)*}e_{(n)}\lambda^{(n+1)} = 1$. To achieve the inductive step, write $i_- = (i_1, \dots, i_n) \in I^n$ whenever $i = (i_1, \dots, i_{n+1}) \in I^{n+1}$, and observe that the following relations hold:

$$\lambda_i^{(n+1)} = \tau^{-n/2} \lambda_{i_-}^{(n)} e_{n-1} \cdots e_0 \lambda_{i_{n+1}} \quad (2)$$

and

$$e_{(n)} = \tau^{-n} e_n e_{n+1} \cdots e_{2n} e_{(n-1)} e_{2n-1} e_{2n-2} \cdots e_n; \quad (3)$$

note that $\lambda_i^{(n)} \in M_{n-1}$ and consequently $\lambda_i^{(n)}$ commutes with e_m for $m \geq n$; finally, compute thus:

$$\begin{aligned} \lambda^{(n+1)*} e_{(n)} \lambda^{(n+1)} &= \sum_{i \in I^{n+1}} \tau^{-2n} \lambda_{i_{n+1}}^* e_0 \cdots e_{n-1} \lambda_{i_-}^{(n)*} e_n \cdots e_{2n} e_{(n-1)} e_{2n-1} \cdots e_n \lambda_{i_-}^{(n)} e_{n-1} \cdots e_0 \lambda_{i_{n+1}} \\ &= \sum_{i_{n+1}} \tau^{-2n} \lambda_{i_{n+1}}^* e_0 \cdots e_{n-1} e_n \cdots e_{2n} \\ &\quad \times \left(\sum_{i_-} \lambda_{i_-}^{(n)*} e_{(n-1)} \lambda_{i_-}^{(n)} \right) e_{2n-1} \cdots e_n e_{n-1} \cdots e_0 \lambda_{i_{n+1}} \\ &= \sum_{i_{n+1}} \tau^{-2n} \lambda_{i_{n+1}}^* e_0 \cdots e_{2n} e_{2n-1} \cdots e_0 \lambda_{i_{n+1}} \\ &= \sum_{i_{n+1}} \lambda_{i_{n+1}}^* e_0 \lambda_{i_{n+1}} = 1, \quad \text{as desired.} \end{aligned}$$

Remark 8. It might be worthwhile to point out here that once one has noticed the validity of Prop. 7(i), the formulae for $\lambda^{(n)}$ and $e_{(n)}$ are arrived at naturally. Consider the case $n = 3$; first, $\tau^{-1/2}e_0\lambda$ is a right basis for M_1/M , so $\tau^{-1}e_1e_0\lambda$ is a right basis for M_2/M_1 ; so, if $x \in M_2$, we may write

$$\begin{aligned} x &= \sum_{i_1} x_{i_1} (\tau^{-1}e_1e_0\lambda_{i_1}) \quad (x_{i_1} \in M_1 \forall i_1) \\ &= \sum_{i_1, i_2} x_{i_2 i_1} (\tau^{-1/2}e_0\lambda_{i_2}) (\tau^{-1}e_1e_0\lambda_{i_1}) \quad (x_{i_2 i_1} \in M) \\ &= \sum_{i_1, i_2, i_3} x_{i_3 i_2 i_1} (\lambda_{i_3}) (\tau^{-1/2}e_0\lambda_{i_2}) (\tau^{-1}e_1e_0\lambda_{i_1}) = \sum_{i \in I^3} x_i \lambda_i^{(3)}. \end{aligned}$$

A second point to be made here is that even if λ is an orthogonal basis in the sense of [5], then $\lambda^{(2)}$ may fail to be an orthogonal basis; this was the reason for relaxing orthogonality and introducing Definition 2.

Finally, to see why the formula for e_n is also to be expected, note first that if $i \vee j = (i_1, \dots, i_n, j_1, \dots, j_m)$ where $i \in I^n$, $j \in I^m$, then we have the following generalization of equation (2):

$$\lambda_{i \vee j}^{(n+m)} = \tau^{-mn/2} \lambda_i^{(n)} (e_{n-1} \cdots e_0) (e_n \cdots e_1) \cdots (e_{m+n-2} \cdots e_{n-1}) \lambda_j^{(m)};$$

the similarity between this expression and the formula $\lambda_{i_1, i_2}^{(2)} = \tau^{-1/2} \lambda_{i_1} e_0 \lambda_{i_2}$ should suggest to the reader that the projection in M_{2n-1} that implements the conditional expectation of M_{n-1} onto N would be of the form

$$\tau^{(\text{some power})} (e_{n-1} \cdots e_0) (e_n \cdots e_1) \cdots (e_{2n-2} \cdots e_{n-1}).$$

□

PROPOSITION 9

Define $\psi_n \in \text{Mat}_{I^n \times I^{n+1}}(N)$ by $\psi_n = E_N(\lambda^{(n)} \lambda^{(n+1)*})$ for $n \geq 0$ (with the convention that $I^0 = \cdot$ and $\lambda^{(0)} = [1]$); also define $\eta_n: M_{n-1} \rightarrow \text{Mat}_{\times I^n}(N)$ by $\eta_n(x) = E_N(x \lambda^{(n)*})$. Then,

- (i) $\eta_{n+1}(x) = \eta_n(x) \psi_n$, for all $x \in M_{n-1}$;
- (ii) $\eta_n(E_{M_{n-1}}(x)) = \eta_m(x) E_N(\lambda^{(m)} \lambda^{(n)*}) = E_N(x \lambda^{(n)*})$, whenever $x \in M_{m-1}$, where $n \leq m$;
- (iii) if $n < m < p$, then

$$E_N(\lambda^{(p)} \lambda^{(n)*}) = E_N(\lambda^{(p)} \lambda^{(m)*}) E_N(\lambda^{(m)} \lambda^{(n)*}),$$

and consequently,

$$E_N(\lambda^{(p)} \lambda^{(n)*}) = \psi_{p-1}^* \psi_{p-2}^* \cdots \psi_n^*.$$

Proof. Observe to start with that

$$x = \eta_n(x) \lambda^{(n)} \quad \text{for all } x \in M_{n-1}$$

(cf. Prop. 5(i) and Prop. 7(iii)).

- (i) $\eta_{n+1}(x) = E_N(x \lambda^{(n+1)*}) = E_N(\eta_n(x) \lambda^{(n)} \lambda^{(n+1)*})$
 $= \eta_n(x) E_N(\lambda^{(n)} \lambda^{(n+1)*}) = \eta_n(x) \psi_n.$
- (ii) $\eta_n(E_{M_{n-1}}(x)) = E_N(E_{M_{n-1}}(x) \lambda^{(n)*}) = E_N(E_{M_{n-1}}(x \lambda^{(n)*}))$
 $= E_N(x \lambda^{(n)*}) = E_N(\eta_m(x) \lambda^{(m)} \lambda^{(n)*})$
 $= \eta_m(x) E_N(\lambda^{(m)} \lambda^{(n)*})$

(iii) Fix $i \in I^p$, then $\lambda_i^{(p)} \in M_{p-1}$ and

$$\begin{aligned} E_N(\lambda_i^{(p)} \lambda^{(n)*}) &= \eta_n(E_{M_{n-1}} \lambda_i^{(p)}) \text{ by (ii)} \\ &= \eta_n(E_{M_{n-1}}(E_{M_{m-1}}(\lambda_i^{(p)}))) \\ &= \eta_m(E_{M_{m-1}}(\lambda_i^{(p)})) E_N(\lambda^{(m)} \lambda^{(n)*}) \text{ by (ii)} \\ &= E_N(\lambda_i^{(p)} \lambda^{(m)*}) E_N(\lambda^{(m)} \lambda^{(n)*}) \end{aligned}$$

and hence

$$E_N(\lambda^{(p)} \lambda^{(n)*}) = E_N(\lambda^{(p)} \lambda^{(m)*}) E_N(\lambda^{(m)} \lambda^{(n)*});$$

it follows that

DEFINITION 10

If $\lambda \in \text{Mat}_{I \times I}(M)$ is a right basis for M/N , we shall let $\theta(=\theta_\lambda)$ denote the map $\theta: N \rightarrow \text{Mat}_{I \times I}(N)$ defined by $\theta(x) = E_N(\lambda x \lambda^*)$. Further, we shall write $\theta^{(n)}$ for $\theta_{\lambda^{(n)}}$ so that $\theta^{(n)}: N \rightarrow \text{Mat}_{I^n \times I^n}(N)$.

PROPOSITION 11

Let $\lambda \in \text{Mat}_{I \times I}(N)$ be a right basis for M/N .

- (i) θ is a faithful normal $*$ -homomorphism such that $\theta(1) = Q_\lambda$ -in particular, if $[M:N]$ is not an integer, then θ is not unital;
- (ii) $\theta_{ij}^{(n)}(x) = \theta_{i_1 j_1}(\theta_{i_2 j_2}(\cdots (\theta_{i_n j_n}(x)) \cdots))$ for all $x \in N$ and $\mathbf{i}, \mathbf{j} \in I^n$, for all n ;
- (iii) $\psi_n \psi_n^* = \theta^{(n)}(1)$ and $\psi_n = \psi_n \theta^{(n+1)}(1)$;
- (iv) $\theta^{(n)}(x) \psi_n = \psi_n \theta^{(n+1)}(x)$, $x \in N$, $n \geq 0$.

Proof. (i) The map θ is clearly linear and σ -weakly continuous; also, it is clear that θ preserves adjoints; finally, if $x, y \in N$,

$$\begin{aligned} \theta(x)\theta(y)e_0 &= E_N(\lambda x \lambda^*)E_N(\lambda y \lambda^*)e_0 = e_0 \lambda x \lambda^* e_0 \lambda y \lambda^* e_0 \\ &= e_0 \lambda x y \lambda^* e_0 = \theta(xy)e_0 \end{aligned}$$

so that θ preserves products.

- (ii) If $\mathbf{i}, \mathbf{j} \in I^{n+1}$ and $x \in N$, use equation (2) to deduce that

$$\begin{aligned} \theta_{ij}^{(n+1)}(x) &= \tau^{-n} E_N(\lambda_{i_-}^{(n)} e_{n-1} \cdots e_0 \lambda_{i_{n+1}} x \lambda_{j_{n+1}}^* e_0 \cdots e_{n-1} \lambda_{j_-}^{(n)*}) \\ &= \tau^{-n} E_N(\lambda_{i_-}^{(n)} e_{n-1} \cdots e_1 \theta_{i_{n+1} j_{n+1}}(x) e_0 \cdots e_{n-1} \lambda_{j_-}^{(n)*}) \\ &= \tau^{-n} E_N(\lambda_{i_-}^{(n)} \theta_{i_{n+1} j_{n+1}}(x) e_{n-1} \cdots e_0 \cdots e_{n-1} \lambda_{j_-}^{(n)*}) \\ &= \tau^{-1} E_N(\lambda_{i_-}^{(n)} \theta_{i_{n+1} j_{n+1}}(x) e_{n-1} \lambda_{j_-}^{(n)*}) \\ &= E_N(\lambda_{i_-}^{(n)} \theta_{i_{n+1} j_{n+1}}(x) \lambda_{j_-}^{(n)*}) \text{ (by (P4)(iii))} \\ &= \theta_{i_- j_-}^{(n)}(\theta_{i_{n+1} j_{n+1}}(x)). \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \psi_n \psi_n^* &= E_N(\lambda^{(n)} \lambda^{(n+1)*}) E_N(\lambda^{(n+1)} \lambda^{(n)*}) \\ &= E_N(E_N(\lambda^{(n)} \lambda^{(n+1)*}) \lambda^{(n+1)} \lambda^{(n)*}) \\ &= E_N(\lambda^{(n)} \lambda^{(n)*}) \text{ (since } \lambda_i^{(n)} \in M_{n-1} \subseteq M_n) \\ &= \theta^{(n)}(1) \end{aligned}$$

while the equation $\psi_n = \psi_n \theta^{(n+1)}(1)$ follows from an entirely analogous computation. (Alternatively, it follows from $\psi_n \psi_n^* = \theta^{(n)}(1)$ that ψ_n is a partial isometry and $\psi_n = \psi_n \psi_n^* \psi_n = \theta^{(n)}(1) \psi_n$, and the equation $\psi_n = \psi_n \theta^{(n+1)}(1)$ is a special case of (iv).)

$$\begin{aligned} \text{(iv)} \quad \theta^{(n)}(x) \psi_n &= E_N(\lambda^{(n)} x \lambda^{(n)*}) E_N(\lambda^{(n)} \lambda^{(n+1)*}) \\ &= E_N(E_N(\lambda^{(n)} x \lambda^{(n)*}) \lambda^{(n)} \lambda^{(n+1)*}) = E_N(\lambda^{(n)} x \lambda^{(n+1)*}); \\ \psi_n \theta^{(n+1)}(x) &= E_N(\lambda^{(n)} \lambda^{(n+1)*}) E_N(\lambda^{(n+1)} x \lambda^{(n+1)*}) \\ &= E_N(E_N(\lambda^{(n)} \lambda^{(n+1)*}) \lambda^{(n+1)} x \lambda^{(n+1)*}) \\ &= E_N(\lambda^{(n)} x \lambda^{(n+1)*}). \end{aligned}$$

□

For the sake of typographical convenience, we shall adopt the classical 'summation convention' regarding covariant and contravariant indices. Hence, a matrix $x \in \text{Mat}_{I \times J}(N)$ will be written $x = ((x_j^i))$, so that if also $y \in \text{Mat}_{J \times K}(N)$, then $xy = ((x_j^i y_k^j))$. (Caution: an index is a 'summing index' only if it appears (at least) once covariantly and (at least) once contravariantly; thus, for instance, $\delta_j^i \xi_j \neq \sum_j \delta_j^i \xi_j$.)

Sometimes, while dealing with expressions involving adjoints, we shall choose to dispense with the summation convention, and shall indicate such a departure by resorting to the familiar symbol $\sum$.

Finally, if λ is a right basis for M/N and if θ, ψ_0, ψ_1 are as before, we shall write:

$$\left. \begin{aligned} \theta(x) &= ((\theta_j^i(x)), \quad \theta^{(2)}(x) = ((\theta_{j_1}^{i_1}(\theta_{j_2}^{i_2}(x)))) \\ \psi_0 &= ((\psi_i)), \quad \psi_1 = ((\psi_{jk})) \text{ and finally,} \\ \psi^i &= \psi_i^*, \quad \psi_i^{jk} = (\psi_{jk}^i)^* \text{ and} \\ \xi^i &= (\xi_i)^* \text{ whenever } \xi = ((\xi_i)) \in \text{Mat}_{\times I}(N) \end{aligned} \right\} \quad (4)$$

This section will be devoted to showing how the pair (M, N) can be recaptured from θ, ψ_0 and ψ_1 by a few 'group-like' axioms.

DEFINITION 1

Let I be a finite set, N a II_1 factor, $\theta: N \rightarrow \text{Mat}_{I \times I}(N)$ a faithful normal *-homomorphism, $\psi_0 \in \text{Mat}_{\times I}(N)$ and $\psi_1 \in \text{Mat}_{I \times I^2}(N)$. Let us write τ for $(\text{Tr } \theta(1))^{-1}$.

The system $\mathcal{G} = (I, N, \theta, \psi_0, \psi_1)$ will be called a 'covariant system' if the following axioms are satisfied:

Identity: $\theta_k^i(\psi_i) \psi_j^{kl} = \tau^{1/2} \theta_j^i(1) = \psi_k \psi_j^{ki} \quad \forall i, j \in I$

Associativity: $\psi_{pq}^i \psi_{kl}^p = \psi_{pq}^i \theta_k^p \psi_l^q \quad \forall i, j, k, l \in I$

Involution: $\theta_p^i(\psi_q \psi_{rk}^q) \psi_j^{pr} = \tau^{1/2} \psi_{jk}^i \quad \forall i, j, k \in I$

Embedding: $\psi_0 \psi_0^* = 1, \quad \psi_1 \psi_1^* = \theta(1)$

Module: $a \psi_0 = \psi_0 \theta(a), \quad \theta(a) \psi_1 = \psi_1 \theta^{(2)}(a) \quad \forall a \in N$

(where of course $\theta^{(2)}: N \rightarrow \text{Mat}_{I^2 \times I^2}(N)$ is related to θ as in Prop. 3.11 (ii)).

Finally we shall write $|\mathcal{G}| = \tau^{-1}$ and call $|\mathcal{G}|$ the order of the covariant system. $\square$

DEFINITION 2

A covariant system $\mathcal{G}$ as in Definition 1 is said to be:

(i) outer, if $\xi \in \text{Mat}_{\times I}(N)$, $\xi = \xi \theta(1)$ and $a \xi = \xi \theta(a)$ for all $a \in N$ imply ξ is a scalar multiple of ψ_0 ; and

(ii) unimodular, if for every $\xi \in \text{Mat}_{\times I}(N)$ satisfying $\xi = \xi \theta(1)$, the following equality holds: $\text{Tr } \xi \xi^* = \tau^{-1} \text{Tr } (\psi_0 \psi_1)_{li} \theta_k^i(\xi^j \xi_l) (\psi_1^* \psi_0^*)^{kl}$.

PROPOSITION 3

Let $(M, N) \in \mathcal{C}$ and let $\lambda \in \text{Mat}_{I \times \cdot}(M)$ be a right basis for M/N . Define θ, ψ_0 and ψ_1 by $\theta(a) = E_N(\lambda a \lambda^*)$, $\psi_0 = E_N(\lambda^*)$, $\psi_1 = E_N(\lambda \lambda^{(2)*})$. Then $\mathcal{G}_\lambda = (I, N, \theta, \psi_0, \psi_1)$ is a unimodular covariant system which is outer precisely when $M \cap N' = \mathbb{C}$.

Proof. The conditions we have chosen to call ‘Embedding’ and ‘Module’ were verified earlier (cf. Prop 3.11 (iii) and (iv)). Before verifying the other axioms, we pause to notice that

$$\begin{aligned}\psi_i &= E_N(\lambda_i^*), \quad \psi_{jk}^i = E_N(\lambda_i(\tau^{-1/2}\lambda_j e_0 \lambda_k)^*) = \tau^{1/2} E_N(\lambda_i \lambda_k^* \lambda_j^*) \\ \psi^i &= E_N(\lambda_i) \text{ and } \psi_{jk}^{i*} = \tau^{1/2} E_N(\lambda_j \lambda_k \lambda_i^*).\end{aligned}$$

Now compute as follows:

$$\begin{aligned}\theta_k^i(\psi_l) \psi_j^{kl} &= \sum_{k,l} E_N(\lambda_i E_N(\lambda_l^*) \lambda_k^*) \tau^{1/2} E_N(\lambda_k \lambda_l \lambda_j^*) \\ &= \tau^{1/2} \sum_{k,l} E_N(E_N(\lambda_i E_N(\lambda_l^*) \lambda_k^*) \lambda_k \lambda_l \lambda_j^*) \\ &= \tau^{1/2} \sum_l E_N(\lambda_i E_N(\lambda_l^*) \lambda_l \lambda_j^*) \\ &= \tau^{1/2} E_N\left(\lambda_i \left(\sum_l E_N(1 \cdot \lambda_l^*) \lambda_l\right) \lambda_j^*\right) \\ &= \tau^{1/2} E_N(\lambda_i \lambda_j^*) = \tau^{1/2} \theta_j^i(1),\end{aligned}$$

and

$$\begin{aligned}\psi_k \psi_j^{ki} &= \tau^{1/2} \sum_k E_N(\lambda_k^*) E_N(\lambda_k \lambda_i \lambda_j^*) = \tau^{1/2} E_N\left(\sum_k E_N(\lambda_k^*) \lambda_k \lambda_i \lambda_j^*\right) \\ &= \tau^{1/2} E_N(\lambda_i \lambda_j^*) = \tau^{1/2} \theta_j^i(1); \\ \psi_{pj}^i \cdot \psi_{kl}^p &= \tau \sum_p E_N(\lambda_i \lambda_j^* \lambda_p^*) E_N(\lambda_p \lambda_l^* \lambda_k^*) = \tau E_N\left(\sum_p E_N(\lambda_i \lambda_j^* \lambda_p^*) \lambda_p \lambda_l^* \lambda_k^*\right) \\ &= \tau E_N(\lambda_i \lambda_j^* \lambda_l^* \lambda_k^*)\end{aligned}$$

while

$$\begin{aligned}\psi_{pq}^i \theta_k^p(\psi_l^q) &= \tau \sum_{p,q} E_N(\lambda_i \lambda_q^* \lambda_p^*) E_N(\lambda_p E_N(\lambda_q \lambda_j^* \lambda_l^*) \lambda_k^*) \\ &= \tau E_N\left(\sum_{p,q} E_N(\lambda_i \lambda_q^* \lambda_p^*) \lambda_p E_N(\lambda_q \lambda_j^* \lambda_l^*) \lambda_k^*\right) \\ &= \tau E_N\left(\sum_q \lambda_i \lambda_q^* E_N(\lambda_q \lambda_j^* \lambda_l^*) \lambda_k^*\right) = \tau E_N(\lambda_i \lambda_j^* \lambda_l^* \lambda_k^*),\end{aligned}$$

where the last step uses the obvious fact that ‘ λ^* is a left-basis for M/N ’; finally,

$$\begin{aligned}\theta_p^i(\psi_q \psi_{rk}^q) \psi_j^{pr} &= \tau \sum_{p,q,r} E_N(\lambda_i E_N(\lambda_q^*) E_N(\lambda_q \lambda_k^* \lambda_r^*) \lambda_p^*) E_N(\lambda_p \lambda_r \lambda_j^*) \\ &= \tau \sum_{p,r} E_N\left(\lambda_i E_N\left(\sum_q E_N(\lambda_q^*) \lambda_q \lambda_k^* \lambda_r^*\right) \lambda_p^*\right) E_N(\lambda_p \lambda_r \lambda_j^*) \\ &= \tau \sum_{p,r} E_N(\lambda_i E_N(\lambda_k^* \lambda_r^*) \lambda_p^*) E_N(\lambda_p \lambda_r \lambda_j^*) \\ &= \tau \sum_r E_N(\lambda_i E_N(\lambda_k^* \lambda_r^*) \lambda_r \lambda_j^*) = \tau E_N(\lambda_i \lambda_k^* \lambda_j^*) = \tau^{1/2} \psi_{jk}^i,\end{aligned}$$

thus establishing that $\mathcal{Q}$ is a covariant system

$(\psi_0\psi_1)_{ij} = \psi_p\psi_{ij}^p = \tau^{1/2} \sum_p E_N(\lambda_p^*)E_N(\lambda_p\lambda_j^*\lambda_i^*) = \tau^{1/2} E_N(\lambda_j^*\lambda_i^*)$; hence,

$$\begin{aligned}
& \tau^{-1} \text{Tr} (\psi_0\psi_1)_{ij} \theta_k^i(\xi^j\xi_l)(\psi_1^*\psi_0^*)^{kl} \\
&= \text{Tr} \sum_{i,j,k,l} E_N(\lambda_j^*\lambda_i^*)E_N(\lambda_i\xi_j^*\xi_l\lambda_k^*)E_N(\lambda_k\lambda_l) \\
&= \text{Tr} \sum_{j,l} E_N(\lambda_j^*\xi_j^*\xi_l\lambda_k^*)E_N(\lambda_k\lambda_l) = \text{Tr} \sum_{j,l} E_N(\lambda_j^*\xi_j^*\xi_l\lambda_l) \\
&= \text{Tr} \sum_{j,l} E_N(\lambda_l\lambda_j^*\xi_j^*\xi_l) = \text{Tr} \sum_{j,l} E_N(\lambda_l\lambda_j^*)\xi_j^*\xi_l \\
&= \text{Tr} \sum_{j,l} \theta_j^l(1)\xi_j^*\xi_l \\
&= \text{Tr} \xi_l\theta_j^l(1)\xi^j = \text{Tr} \xi_j\xi^j \text{ (since } \xi = \xi\theta(1)) \\
&= \text{Tr} \xi\xi^*, \text{ as desired.}
\end{aligned}$$

Finally, if we write $\eta(x) = E_N(x\lambda^*)$ for x in M , note then that if $x \in M$ and $a \in N$, then $\eta(ax) = a\eta(x)$ while

$$\eta(xa) = E_N(xa\lambda^*) = E_N(\eta(x)\lambda a\lambda^*) = \eta(x)\theta(a);$$

the last assertion of the proposition follows at once. $\square$

Remark. If α is an outer action of a finite group G on N and if $M = N \rtimes_\alpha G$, the natural right basis $\lambda \in \text{Mat}_{G \times \times}(M)$ for M/N is such that $s \mapsto \lambda_s$ is a group homomorphism of G into the unitary group of M such that $\lambda_s a \lambda_{s^{-1}} = \alpha_s(a)$ for all $s \in G$ and $a \in M$. The θ, ψ_0 and ψ_1 of the associated $\mathcal{G}_\lambda$ are given by $\theta_i^s(a) = \delta_{is}^s \alpha_s(a)$, $\psi_s = \delta_{ss}$, where ε denotes the identity element of G , and

$$\psi_{tu}^s = \begin{cases} |G|^{-1/2}, & \text{if } s = tu \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if $\alpha: G \rightarrow \text{Aut } N_0$ (N_0 a II_1 factor) is an outer action of a countable group G on N_0 , and if H is a subgroup of finite index in G , a right basis for $N_0 \rtimes G/N_0 \rtimes H$ is given by $\lambda = ((\lambda_{s_i}))$ where $\{s_i\}$ is a set of coset representatives, i.e., $G = \text{II}_i H s_i$. The associated covariant system $\mathcal{G}_\lambda$ is seen to be given by:

$$\begin{aligned}
\theta_j^i \left(\sum_{h \in H} x_h \lambda_h \right) &= \sum_{\{h: s_i h s_j^{-1} \in H\}} \alpha_{s_i}(x_h) \lambda_{s_i h s_j^{-1}}, \\
\psi_i &= \begin{cases} 0, & \text{if } s_i \notin H \\ \lambda_{s_i}, & \text{if } s_i \in H \end{cases}
\end{aligned}$$

and

$$\psi_{jk}^i = \begin{cases} 0 & \text{if } s_j s_k \notin H s_i \\ [G:H]^{-1/2} \lambda_{s_j s_k^{-1} s_i^{-1}}, & \text{if } s_j s_k \in H s_i. \end{cases} \quad \square$$

Lemma 4. Let $\mathcal{G} = (I, N, \theta, \psi_0, \psi_1)$ be a covariant system. Then

- (i) $\psi_0 = \psi_0\theta(1)$; (ii) $\psi^{ij} = \psi^{ij}\theta(1)$ (i.e. $\psi_k^{ij} = \psi_p^{ij}\theta_k^p(1)$) $\forall i, j \in I$; (iii) $\theta_i^j(a) = \theta_i^j(a)\theta(1)$, $\forall i \in I$.

Proof. (i) follows at once from the (Module) axiom (put $a = 1$), while (iii) follows from the fact that θ is a homomorphism. As for (ii), the embedding axiom $\psi_1 \psi_1^* = \theta(1)$ implies that $\psi_1^* = \psi_1^* \theta(1)$, which is the content of the parenthetical remark in (ii). $\square$

DEFINITION 5

If $\mathcal{G}$ is a covariant system, define $\mathcal{U} = \{\xi \in \text{Mat}_{\times I}(N) : \xi = \xi \theta(1)\}$. If $\xi, \eta \in \mathcal{U}$, define

$$(\xi \cdot \eta)_i = \tau^{-1/2} \xi_j \theta_k^i(\eta_l) \psi_l^{kl},$$

and

$$(\xi^b)_i = \tau^{-1/2} \psi_j \psi_{kl}^i \theta_i^k(\xi^l). \quad \square$$

It follows from Lemma 4 that if $\xi, \eta \in \mathcal{U}$, then $\xi \cdot \eta = (((\xi \cdot \eta)_i))$ and $\xi^b = (((\xi^b)_i))$ both belong to $\mathcal{U}$. It is easily verified, further, that if $\xi \in \mathcal{U}$ and if $a, b \in N$, then the matrix-product $a \xi \theta(b)$ also belongs to $\mathcal{U}$; thus $\mathcal{U}$ has a natural N -bimodule structure.

Lemma 6. The operations defined above endow $\mathcal{U}$ with the structure of an involutive associative algebra, and ψ_0 is the (multiplicative) identity for $\mathcal{U}$.

Proof. Clearly $\mathcal{U}$ is a vector space; further, if $\xi \in \mathcal{U}$, then for any $i \in I$,

$$\begin{aligned} (\xi \cdot \psi_0)_i &= \tau^{-1/2} \xi_j \theta_k^i(\psi_l) \psi_l^{kl} = \xi_j \theta_i^j(1) \text{ (by 'Identity')} \\ &= \xi_i; \\ (\psi_0 \cdot \xi)_i &= \tau^{-1/2} \psi_j \theta_k^i(\xi_l) \psi_l^{kl} = \tau^{-1/2} \xi_l \psi_k \psi_l^{kl} \text{ (by 'Module')} \\ &= \xi_l \theta_i^l(1) = \xi_i. \end{aligned}$$

Next, let $\xi, \eta, \zeta \in \mathcal{U}$ and $i \in I$; then

$$\begin{aligned} ((\xi \cdot \eta) \cdot \zeta)_i &= \tau^{-1/2} (\xi \cdot \eta)_r \theta_s^i(\zeta_t) \psi_t^{st} = \tau^{-1} \xi_j \theta_k^i(\eta_l) \psi_r^{kl} \theta_s^i(\zeta_t) \psi_t^{st} \\ &= \tau^{-1} \xi_j \theta_k^i(\eta_l) \theta_{pq}^{(2)kl}(\zeta_t) \psi_s^{pq} \psi_t^{st} \text{ (by 'Module')} \\ &= \tau^{-1} \xi_j \theta_k^i(\eta_l) \theta_{pq}^{(2)kl}(\zeta_t) \theta_r^p(\psi_s^{st}) \psi_t^{rs} \text{ (by 'Associativity')} \\ &= \tau^{-1} \xi_j \theta_k^i(\eta_l \theta_q^l(\zeta_t) \psi_s^{st}) \psi_t^{rs} \\ &= \tau^{-1/2} \xi_j \theta_r^i((\eta \cdot \zeta)_s) \psi_t^{rs} = (\xi \cdot (\eta \cdot \zeta))_i, \end{aligned}$$

so $\mathcal{U}$ is an associative unital algebra.

Now for the involution; if $\xi \in \mathcal{U}$,

$$\begin{aligned} ((\xi^b)^b)_i &= \tau^{-1/2} \psi_r \psi_{st}^r \theta_i^s((\xi^b)^t) = \tau^{-1} \psi_r \psi_{st}^r \theta_i^s(\theta_k^t(\xi_l)) \psi_j^{kl} \psi_j^{kj} \\ &= \tau^{-1} \psi_r \psi_{st}^r \theta_{pk}^{(2)st}(\xi_l) \theta_i^p(\psi_j^{kl} \psi_j^{kj}) = \tau^{-1} \xi_l \psi_r \psi_{pk}^r \theta_i^p(\psi_j^{kl} \psi_j^{kj}) \text{ (by 'Module')} \\ &= \tau^{-1/2} \xi_l \psi_r \psi_{pk}^r \theta_i^l \text{ (by 'Involution')} \\ &= \xi_l \theta_i^l(1) \text{ (by 'Identity')} \\ &= \xi_i \text{ (since } \xi \in \mathcal{U}); \end{aligned}$$

and if $\xi, \eta \in \mathcal{U}$, then

$$\begin{aligned}
 (\eta^b \cdot \xi^b)_i &= \tau^{-1/2} (\eta^b)_j \theta_k^j((\xi^b)_l) \psi_i^{kl} \\
 &= \tau^{-3/2} \psi_m \psi_{np}^m \theta_j^n(\eta^p) \theta_k^j(\psi_r \psi_{st}^r \theta_l^s(\xi^t)) \psi_i^{kl} \\
 &= \tau^{-3/2} \psi_m \psi_{np}^m \theta_j^n(\eta^p) \theta_q^j(\psi_r \psi_{st}^r) \theta_{kl}^{(2)qs}(\xi^t) \psi_i^{kl} \\
 &= \tau^{-3/2} \psi_m \psi_{np}^m \theta_j^n(\eta^p) \theta_q^j(\psi_r \psi_{st}^r) \psi_k^{qs} \theta_l^s(\xi^t) \quad (\text{by 'Module'}) \\
 &= \tau^{-1} \psi_m \psi_{np}^m \theta_j^n(\eta^p) \psi_k^j \theta_l^k(\xi^t) \quad (\text{by 'Involution'}) \\
 &= \tau^{-1} \psi_m \psi_{np}^m \psi_{rs}^n \theta_{kl}^{(2)rs}(\eta^p) \theta_i^k(\xi^t) \quad (\text{by 'Module'}) \\
 &= \tau^{-1} \psi_m \psi_{iq}^m \theta_l^i(\psi_{ps}^q \theta_{kr}^{(2)rs}(\eta^p) \theta_l^k(\xi^t)) \quad (\text{by 'Associativity'}) \\
 &= \tau^{-1} \psi_m \psi_{iq}^m \theta_l^i(\psi_{ps}^q \theta_{kr}^{(2)rs}(\eta^p) \xi^t) \\
 &= \tau^{-1/2} \psi_m \psi_{iq}^m \theta_l^i((\xi \cdot \eta)^q) = ((\xi \cdot \eta)^b)_i,
 \end{aligned}$$

and the proof is complete. $\square$

Lemma 7. Let $\mathcal{K} = \text{Mat}_{\times I}(L^2(N))$ be equipped with inner product $(\xi, \eta) = \sum_i (\xi_i, \eta_i)$ and let $\mathcal{H} = \{\xi \in \mathcal{K} : \xi = \xi \theta(1)\}$.

- (i) $\mathcal{K}$ is a Hilbert space and $\mathcal{H}$ is a closed subspace of $\mathcal{K}$;
- (ii) the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ is given by $\xi \mapsto \xi \theta(1)$;
- (iii) the following equations define bounded operators on $\mathcal{H}$:

$$\begin{aligned}
 L_a \xi &= a \xi, \quad a \in N \\
 R_{\theta(a)} \xi &= \xi \theta(a), \quad a \in N \\
 R_x \xi &= \xi x,
 \end{aligned}$$

where x is any element of $\text{Mat}_{I \times I}(N)$ satisfying $\theta(1)x\theta(1) = \theta(1)x$.

Proof. The first two assertions are clear while the last is a consequence of the fact that if $S \in \mathcal{L}(\mathcal{K})$ and $S(\mathcal{H}) \subseteq \mathcal{H}$, then $S|_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$, and the fact that $L^2(N)$ is a N -bimodule. $\square$

Lemma 8. The equation $E\xi = \xi \psi_0^*$ defines a bounded operator from $\mathcal{H}$ to $L^2(N)$ such that $E(\mathcal{U}) \subseteq N$; further, for any $\xi, \eta \in \mathcal{U}$,

$$(\xi, \eta) = \text{Tr } E(\xi \cdot \eta^b).$$

Proof. It is clear that E is continuous and that $E(\mathcal{U}) \subseteq N$; finally if $\xi, \eta \in \mathcal{U}$, then

$$\begin{aligned}
 E(\xi \cdot \eta^b) &= (\xi \cdot \eta^b)_i \psi^i = \tau^{-1} \xi_j \theta_k^j((\psi_0 \psi_1)_{qr} \theta_l^q(\eta^r)) \psi_i^{kl} \psi^i \\
 &= \tau^{-1} \xi_j \theta_m^j((\psi_0 \psi_1)_{qr}) \theta_{kl}^{(2)mq}(\eta^r) \psi_i^{kl} \psi^i \\
 &= \tau^{-1} \xi_j \theta_m^j((\psi_0 \psi_1)_{qr}) \psi_k^{mq} \psi^k \eta^r \quad (\text{by 'Module'}) \\
 &= \tau^{-1/2} \xi_j \psi_{kr}^j \psi^k \eta^r \quad (\text{by 'Involution'})
 \end{aligned}$$

Thus

$$E(\xi \cdot \eta^b) = \xi_j \eta^j \quad \text{for } \xi, \eta \in \mathcal{U} \quad (5)$$

so that, in particular, $\text{Tr } E(\xi \cdot \eta^b) = (\xi, \eta)$. $\square$

PROPOSITION 9

Let $\mathcal{G}$ be any covariant system and let $\mathcal{U}$ be constructed as above. Then $\mathcal{U}$ has the structure of a right Hilbert algebra whose associated modular operator is bounded. The modular operator is the identity if and only if $\mathcal{G}$ is unimodular.

Proof. It is clear from the definition of the product in $\mathcal{U}$ that $\xi \cdot \eta = R_y \xi$ (in the notation of Lemma 7(iii)) where $y_j^i = \tau^{-1/2} \theta_k^i(\eta_j) \psi_j^{k^i}$; hence $\pi_r(\eta) = R_y$. Also, if $\xi, \eta, \zeta \in \mathcal{U}$, then by Lemma 8,

$$\langle \xi \cdot \eta, \zeta \rangle = \text{Tr } E((\xi \cdot \eta) \cdot \zeta^b) = \text{Tr } E(\xi \cdot (\zeta \cdot \eta^b)^b) = \langle \xi, \zeta \cdot \eta^b \rangle.$$

We need, now, to estimate $\|\xi^b\|_{\mathcal{H}}$; for this, first deduce from the embedding condition that $\psi_0 \psi_1$ is a partial isometry and hence $\|\tilde{\xi} \psi_1^* \psi_0^* \|_{L^2(N)} \leq \|\tilde{\xi}\|_{\text{Mat.} \times L^2(L^2(N))}$ whenever $\tilde{\xi} \in \text{Mat.} \times L^2(L^2(N))$.

Now, if $\xi \in \mathcal{U}$, then for each $i \in I$, define $\tilde{\xi}^{(i)} \in \text{Mat.} \times L^2(L^2(N))$ by $\tilde{\xi}_{jk}^{(i)} = \theta_j^i(\xi_k)$ and notice that

$$\tilde{\xi}^{(i)} \psi_1^* \psi_0^* = \theta_j^i(\xi_k) (\psi_1^* \psi_0^*)^{jk} = (\tau^{1/2} \xi_i^b)^*;$$

hence, for any $\xi \in \mathcal{U}$,

$$\begin{aligned} \|\xi^b\|_{\mathcal{H}}^2 &= \sum_i \|\xi_i^b\|_{L^2(N)}^2 \leq \tau^{-1} \sum_i \|\tilde{\xi}^{(i)}\|^2 = \tau^{-1} \sum_{i,j,k} \|\theta_j^i(\xi_k)\|_{L^2(N)}^2 \\ &= \tau^{-1} \text{Tr } \theta_j^i(\xi_k) \theta_i^j(\xi_k) = \tau^{-1} \text{Tr } \theta_i^i(\xi_k \xi_k) = \tau^{-1} \text{Tr } \theta(\xi_k \xi_k) \\ &= \tau^{-2} \text{Tr } \xi_k \xi_k = \tau^{-2} \|\xi\|_{\mathcal{H}}^2, \end{aligned}$$

where we used the uniqueness of the trace on N to conclude that $\text{Tr } \theta(a) = \tau^{-1} \text{Tr } (a)$ for every $a \in N$.

An easy computation shows that for any $\xi \in \mathcal{U}$,

$$\|\xi^b\|^2 = \tau^{-1} \text{Tr } (\psi_0 \psi_1)_{ij} \theta_k^i(\xi^j \xi_i) (\psi_1^* \psi_0^*)^{k^i},$$

so that the second assertion of the Proposition is a consequence of the definitions. $\square$

Lemma 10. The equation $\lambda_j^{(i)} = \theta_j^i(1)$ defines an element $\lambda^{(i)}$ of $\mathcal{U}$ such that $E(\xi \cdot \lambda^{(i)b}) = \xi^i$ for all $\xi \in \mathcal{U}$.

Proof. That $\lambda^{(i)} \in \mathcal{U}$ follows from Lemma 4(iii); as for the second assertion, if $\xi \in \mathcal{U}$, then it follows from equation (5) that

$$E(\xi \cdot \lambda^{(i)b}) = \xi_j \lambda^{(i)j} = \xi_j \theta_j^i(1) = \xi_i \quad \square$$

Lemma 11. Suppose ξ is a left bounded element of $\mathcal{H}$: i.e., suppose there is a constant $K > 0$ such that $\|\pi(n)\xi\| \leq K\|n\|$ for all $n \in \mathcal{U}$. Then $E\xi \in N$ and $\|E\xi\| \leq \|\pi(\xi)\|$.

Proof. Since the left bounded elements of $L^2(N)$ are precisely the elements of N , both assertions of the Lemma are consequences of the following identity which we shall establish:

$$\langle \pi_i(\xi) a \psi_0, b \psi_0 \rangle_{\mathcal{H}} = \langle (E\xi) a, b \rangle_{L^2(N)}, \text{ for all } a, b \in N.$$

So, let $a, b \in N$, and compute:

$$\begin{aligned} \langle \pi_i(\xi) a \psi_0, b \psi_0 \rangle_{\mathcal{H}} &= \langle \pi_i(a \psi_0) \xi, b \psi_0 \rangle_{\mathcal{H}} = \sum_i \tau^{-1/2} \langle \xi_j \theta_k^i(a \psi_1) \psi_i^{kl}, b \psi_1 \rangle_{L^2(N)} \\ &= \sum_i \tau^{-1/2} \langle \xi_j \theta_p^i(a) \theta_k^p(\psi_1) \psi_i^{kl}, b \psi_1 \rangle_{L^2(N)} \\ &= \sum_i \langle \xi_j \theta_p^i(a) \theta_p^i(1), b \psi_1 \rangle_{L^2(N)} \\ &= \sum_i \langle \xi_j \theta_i^j(a), b \psi_1 \rangle_{L^2(N)} = \sum_j \langle \xi_j, b \psi_1 \theta_j^i(a^*) \rangle_{L^2(N)} \\ &= \sum_j \langle \xi_j, b a^* \psi_j \rangle_{L^2(N)} = \langle \xi_j \psi^j, b a^* \rangle_{L^2(N)} = \langle E\xi, b a^* \rangle_{L^2(N)} \\ &= \langle (E\xi) a, b \rangle_{L^2(N)}, \text{ as desired.} \quad \square \end{aligned}$$

Theorem 12. Let $\mathcal{G}$ be any covariant system, let $\mathcal{U}$ be constructed as above, and let $\tilde{M} = \pi_1(\mathcal{U})$, $\tilde{N} = \{\pi_1(a \psi_0) : a \in N\}$. Then $\tilde{M}$ is a finite von Neumann algebra and $\tilde{N}$ is a II_1 subfactor of $\tilde{M}$. Further, $(\tilde{M}, \tilde{N}) \in \mathcal{C}_0$ if and only if $\mathcal{G}$ is outer (in the sense of Definition 2).

Proof. If $a \in N$ and if $\xi \in \mathcal{U}$, then

$$\begin{aligned} (\pi_1(a \psi_0) \xi)_i &= \tau^{-1/2} a \psi_j \theta_k^i(\xi_1) \psi_i^{kl} \\ &= \tau^{-1/2} a \xi_1 \psi_k \psi_i^{kl} \text{ (by 'Module')} \\ &= a \xi_1 \theta_i^k(1) \text{ (by 'Identity')} \\ &= a \xi_i \end{aligned}$$

and hence $\pi_1(a \psi_0) = L_a$ (cf. Lemma 7(iii)). It is clear that $a \mapsto L_a$ is a normal $*$ -homomorphism of N onto $\tilde{N}$; since $L_1 = 1 \neq 0$, and since N is a factor, it follows that $a \mapsto L_a$ is a $*$ -isomorphism of N onto $\tilde{N}$ and that $\tilde{N}$ is a II_1 factor.

Also, it follows directly from Proposition 9 and the general theory of Hilbert algebras that $\pi_1(\mathcal{U})''$ is a finite von Neumann algebra. We shall show now that $\pi_1(\mathcal{U})'' = \pi_1(\mathcal{U})$. So, let $x \in \pi_1(\mathcal{U})''$; since $\pi_*(\mathcal{U}) \subseteq (\pi_1(\mathcal{U})'')'$, it follows that for any $\eta \in \mathcal{U}$, we have

$$x\eta = x\pi_*(\eta)\psi_0 = \pi_*(\eta)x\psi_0;$$

hence $\xi = x\psi_0$ is a left bounded element of $\mathcal{H}$ and $x = \pi_1(\xi)$. With $\lambda^{(i)}$ is in Lemma 10, note that $\lambda^{(i)b} \in \mathcal{U}$ so that $\lambda^{(i)b}$ is left-bounded and consequently $\pi_1(\xi)\lambda^{(i)b}$ is also left-bounded. The same computation as in Lemma 10 shows then that $\xi_i = E(\pi_1(\xi)\lambda^{(i)b})$; an appeal to Lemma 11 shows now that $\xi_i \in N$. Since i was arbitrary, deduce that $\xi \in \mathcal{U}$ and conclude, finally, that $x \in \pi_1(\mathcal{U})$.

Finally, if $x = \pi_1(\xi)$, $\xi \in \mathcal{U}$, an easy computation shows that $x \in \tilde{M} \cap \tilde{N}'$ if and only

if $a\xi = \xi\theta(a)$ for all $a \in N$, and hence $\tilde{M} \cap \tilde{N}' = \mathbf{C1}$ if and only if the covariant system $\mathcal{G}$ is outer. If $\mathcal{G}$ is outer, then $\tilde{M} \cap \tilde{N}' = \mathbf{C1}$ and in particular, $\tilde{M}$ is a factor. Since $\tilde{M}$ is finite and since $\tilde{M}$ contains a Π_1 -subfactor, it must be the case that $\tilde{M}$ is a Π_1 factor. Also, notice that $\pi_l(\xi) = \sum_i \pi_l(\xi_i \psi_0) \pi_l(\lambda^{(i)})$, with $\lambda^{(i)}$ as in Lemma 10; hence $\tilde{M}$ is finitely generated as a left $\tilde{N}$ -module and consequently $[\tilde{M} : \tilde{N}] < \infty$, and the theorem is proved. $\square$

We now proceed to show that every outer covariant system is automatically unimodular. We continue to let $\mathcal{G}$ denote an arbitrary covariant system, and to let $\mathcal{U}$, $\tilde{M}$, $\tilde{N}$ be as in Theorem 12.

Lemma 13. The equation

$$\varphi(\pi_l(\xi)) = \langle \xi, \psi_0 \rangle, \quad \xi \in \mathcal{U}$$

defines a faithful normal state φ on $\tilde{M}$ such that $\varphi(ax) = \varphi(xa)$ for every $a \in \tilde{N}$ and $x \in \tilde{M}$; in particular, $\tilde{N} \subseteq \tilde{M}^\varphi$.

Proof. Since ψ_0 is the identity for $\mathcal{U}$, we see that $\varphi(\pi_l(\xi)) = \langle \pi_l(\xi) \psi_0, \psi_0 \rangle$ and consequently φ is normal and positive. Also $\varphi(\pi_l(\psi_0)) = \langle \psi_0, \psi_0 \rangle = 1$ (since $\psi_i \psi^i = 1$), so that φ is a state; finally, φ is faithful since

$$\varphi(\pi_l(\xi) * \pi_l(\xi)) = \langle \xi, \xi \rangle = \|\xi\|^2.$$

Recall that a typical element of $\tilde{M}$ (resp., $\tilde{N}$) is of the form $\pi_l(\xi)$ (resp., $\pi_l(a\psi_0) = L_a$) where $\xi \in \mathcal{U}$ (resp., $a \in N$); also, notice that for ξ in $\mathcal{U}$ and a in N ,

$$\begin{aligned} (\xi \cdot a \psi_0)_i &= \tau^{-1/2} \xi_j \theta_k^j(a \psi_0) \psi_i^{ki} = \tau^{-1/2} \xi_j \theta_p^j(a) \theta_k^p(\psi_0) \psi_i^{ki} \\ &= \xi_j \theta_p^j(a) \theta_p^i(1) \text{ (by 'Identity')} \\ &= \xi_j \theta_i^j(a); \end{aligned}$$

i.e.,

$$\xi \cdot a \psi_0 = \xi \theta(a).$$

Hence,

$$\begin{aligned} \varphi(\pi_l(a \psi_0) \pi_l(\xi)) &= \langle a \psi_0 \cdot \xi, \psi_0 \rangle = \langle L_a \xi, \psi_0 \rangle \\ &= \langle \xi, L_{a^*} \psi_0 \rangle = \langle \xi, a^* \psi_0 \rangle = \langle \xi, \psi_0 \theta(a^*) \rangle \\ &= \langle \xi \theta(a), \psi_0 \rangle = \langle \xi \cdot a \psi_0, \psi_0 \rangle = \varphi(\pi_l(\xi) \pi_l(a \psi_0)). \end{aligned}$$

The final assertion is a consequence of the well-known characterization of $\tilde{M}^\varphi$ from [4]. $\square$

PROPOSITION 14

An outer covariant system is automatically unimodular.

Proof. It is easy to see that $\mathcal{U}$ is unimodular if and only if the state φ (of Lemma 13) is a trace. To see that φ is a trace, appeal first to the Radon–Nikodym theorem of [4] to deduce the existence of a (possibly unbounded) positive self-adjoint operator h affiliated to $\tilde{M}$ such that $\varphi = \text{Tr}(h \cdot)$. (Recall from Theorem 12 that the ‘outer-ness’

The proposition will be proved once we establish that $h\eta \tilde{M} \cap \tilde{N}'$; for this, appeal once again to [4] to observe that the modular group σ^φ is given by $\sigma^\varphi(x) = h^u \sigma^r(x) h^{-u} = h^u x h^{-u}$; infer now from Lemma 13 that $h^u \in \tilde{M} \cap \tilde{N}'$ for all t , whence $h\eta \tilde{M} \cap \tilde{N}'$. $\square$

In order to complete the discussion of the relationship between outer covariant system and objects of $\mathcal{C}_0$, we need the following definitions.

DEFINITION 15

(a) Two objects (M, N) and $(\tilde{M}, \tilde{N})$ of $\mathcal{C}_0$ are said to be isomorphic if there is a $*$ -isomorphism π of M onto $\tilde{M}$ such that $\pi(N) = \tilde{N}$; in such a case, we shall write $\pi: (M, N) \xrightarrow{\cong} (\tilde{M}, \tilde{N})$; (b) Two covariant systems $\mathcal{G} = (I, N, \theta, \psi_0, \psi_1)$ and $\tilde{\mathcal{G}} = (\tilde{I}, \tilde{N}, \tilde{\theta}, \tilde{\psi}_0, \tilde{\psi}_1)$ are said to be isomorphic if there exist a $*$ -isomorphism π of N onto $\tilde{N}$ and a matrix $u \in \text{Mat}_{I \times \tilde{I}}(\tilde{N})$ such that

- (i) $u^*u = \tilde{\theta}(1)$ and $uu^* = \pi(\theta(1))$;
- (ii) $\pi(\theta(a)) = u\tilde{\theta}(\pi(a))u^*$;
- (iii) $\pi(\psi_0) = \tilde{\psi}_0 u^*$; and
- (iv) $\pi(\psi_{jk}) = \sum_{p,q,r,s \in \tilde{I}} u_{ip} \tilde{\psi}_{qs} \tilde{\theta}_r^s (u_{jq}^* u_r^*)$,

where, in (ii) and (iii) (and in the sequel), we have used the notation $\pi(x) = ((\pi(x_{ij})))$ for any matrix $x = ((x_{ij}))$ with entries in N . $\square$

DEFINITION 16

If $\mathcal{G}$ is a covariant system, and if $\tilde{M}$ and $\tilde{N}$ are as in Theorem 12, we shall call $\tilde{M}$ the 'crossed-product of N by $\mathcal{G}$ ' and write $\tilde{M} = N \times \mathcal{G}$; also, we shall identify N with $\tilde{N}$ with $\tilde{N} (a \mapsto \pi_i(a)\psi_0)$ and think of N as a subfactor of $N \times \mathcal{G}$. $\square$

PROPOSITION 17

(a) Suppose $(M, N), (\tilde{M}, \tilde{N}) \in \mathcal{C}_0$ and suppose $(M, N) \cong (\tilde{M}, \tilde{N})$. Let $\lambda \in \text{Mat}_{I \times I}(M)$ (resp., $\tilde{\lambda} \in \text{Mat}_{\tilde{I} \times \tilde{I}}(\tilde{M})$) be a right basis for M/N (resp., $\tilde{M}/\tilde{N}$), and let $\mathcal{G}_\lambda$ (resp., $\mathcal{G}_{\tilde{\lambda}}$) be the covariant system constructed as in Proposition 3. Then $\mathcal{G}_\lambda \cong \mathcal{G}_{\tilde{\lambda}}$.

(b) Let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be isomorphic covariant systems, with the isomorphism being implemented by (π, u) as in Definition 15(b); then π extends to an isomorphism of $N \times \mathcal{G}$ onto $\tilde{N} \times \tilde{\mathcal{G}}$.

Proof. (a) Suppose $\pi: (M, N) \cong (\tilde{M}, \tilde{N})$. Clearly, then, $\pi(E_N(x)) = E_{\tilde{N}}(\pi(x))$ for x in M ; it follows (from (ii) of Lemma 1.1) that $\pi(\lambda)$ is a right basis for $\tilde{M}/\tilde{N}$. Appeal to Proposition 1.6 to find a partial isometry $u \in \text{Mat}_{I \times \tilde{I}}(\tilde{N})$ such that $u^*u = E_{\tilde{N}}(\tilde{\lambda}\lambda^*)$, and $uu^* = E_{\tilde{N}}(\pi(\lambda\lambda^*))$. A routine computation now shows that the pair (π, u) implements an isomorphism $\mathcal{G}_\lambda \cong \mathcal{G}_{\tilde{\lambda}}$ as in Definition 15(b). (b) Conversely, if (π, u) implements an isomorphism $\mathcal{G} \cong \tilde{\mathcal{G}}$ as in Definition 15(b), it is relatively painless to verify that the map $\xi \mapsto \pi(\xi)u$ defines an isomorphism $\mathcal{U} \cong \tilde{\mathcal{U}}$ of right Hilbert algebras, which in turn yields an isomorphism $(N \times \mathcal{G}, N) \cong (\tilde{N} \times \tilde{\mathcal{G}}, \tilde{N})$. $\square$

Theorem 18. (a) If $(M, N) \in \mathcal{C}_0$, if λ is a right basis for M/N and if $\mathcal{G}_\lambda$ is the associated covariant system (as in Proposition 3), then $(N \times \mathcal{G}_\lambda, N) \cong (M, N)$. (b) If $\mathcal{G}$ is an outer

covariant system, if λ is any right basis for $N \times \mathcal{G}/N$ and if $\mathcal{G}_\lambda$ is the associated covariant system, then $\mathcal{G} \cong \mathcal{G}_\lambda$.

Proof. (a) An isomorphism $\pi: (N \times \mathcal{G}_\lambda, N) \xrightarrow{\cong} (M, N)$ is given by $\pi(\pi_i(\xi)) = \sum \xi_i \lambda_i$. (b) Begin by appealing to Proposition 14 and Lemma 13 to note that the trace on $N \times \mathcal{G}$ is given by φ . Note next, that by definition, $\varphi(\pi_i(\xi)) = \text{Tr } E(\xi)$. Via the identification $N \leftrightarrow \tilde{N}$, we deduce that the restriction of E to $N \times \mathcal{G}$ is the trace-preserving conditional expectation of $N \times \mathcal{G}$ onto N .

Next, note that by Prop. 17(a), it suffices to prove the assertion for any one right basis for $N \times \mathcal{G}/N$. For this purpose, define $\lambda \in \text{Mat}_{I \times I}(N \times \mathcal{G})$ by $\lambda_i = \pi_i(\lambda^{(i)})$, with $\lambda^{(i)}$ as in Lemma 10. We need, first, to verify that $\lambda^* e_0 \lambda = 1$ where e_0 is the orthogonal projection of $\mathcal{H}$ onto $[a\psi_0: a \in N]$.

If $\xi \in \mathcal{H}$ and $a \in N$, note that

$$\begin{aligned} \langle \xi, a\psi_0 \rangle_* &= \sum_i \langle \xi_i, a\psi_i \rangle_{L^2(N)} = \langle \xi_i \psi^i, a \rangle_{L^2(N)} \\ &= \langle \xi \psi_0^*, a \rangle_{L^2(N)} = \langle \xi \psi_0^* \psi_0, a\psi_0 \rangle_* \end{aligned}$$

(since $\langle \xi_0 \psi_0, \eta_0 \psi_0 \rangle_* = \langle \xi_0, \eta_0 \rangle_{L^2(N)}$ for $\xi_0, \eta_0 \in L^2(N)$), and hence $e_0 \xi = \xi \psi_0^* \psi_0$; thus, $(e_0 \xi)_i = \xi_r \psi^r \psi_i$, $\xi \in \mathcal{H}$, $i \in I$.

Now fix $\xi \in \mathcal{H}$ and compute:

$$\begin{aligned} (e_0 \lambda_i \xi)_l &= (e_0(\lambda^{(i)} \cdot \xi))_l = (\lambda^{(i)} \cdot \xi)_r \psi^r \psi_l = \tau^{-1/2} \lambda_p^{(i)} \theta_q^p(\xi_s) \psi_r^{qs} \psi^r \psi_l \\ &= \tau^{-1/2} \theta_p^i(1) \theta_q^p(\xi_s) \psi_r^{qs} \psi^r \psi_l = \tau^{-1/2} \theta_q^i(\xi_s) \psi_r^{qs} \psi^r \psi_l; \end{aligned}$$

hence,

$$\begin{aligned} \theta_k^j(e_0 \lambda_i \xi)_l \psi_l^{kl} &= \tau^{-1/2} \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \theta_k^n(\psi) \psi_l^{kl} \\ &= \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \theta_n^m(1) \text{ (by 'Identity')} \\ &= \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r); \end{aligned}$$

also, in view of Proposition 14, note that the Hilbert algebra $\mathcal{H}$ is unimodular so that $\lambda_i^* = \pi_i(\lambda^{(i)})^* = \pi_i(\lambda^{(i)b})$; hence,

$$\begin{aligned} \left(\sum_i \lambda_i^* e_0 \lambda_i \xi \right)_l &= \sum_i (\lambda^{(i)b} \cdot e_0 \lambda_i \xi)_l \\ &= \sum_i \tau^{-1/2} (\lambda^{(i)b})_j \theta_k^j((e_0 \lambda_i \xi)_l) \psi_l^{kl} \\ &= \sum_i \tau^{-1/2} (\lambda^{(i)b})_j \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \\ &= \sum_i \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_j^k(\lambda^{(i)b}) \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_{ji}^{(2)kl}(1) \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_{mq}^{(2)kl}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} \xi_s (\psi_0 \psi_1)_{mq} \theta_n^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} \xi_s \psi_j \psi_{mq}^j \theta_n^m(\psi_r^{qs} \psi^r) \end{aligned}$$

$$\begin{aligned}
&= \tau^{-1/2} \xi_s \psi_j \psi_t^{js} \text{ (by 'Involution')} \\
&= \xi_s \theta_i^s(1) \text{ (by 'Identity')} \\
&= \xi_i;
\end{aligned}$$

i.e., $\sum_i \lambda_i^* e_0 \lambda_i = 1$.

Hence λ is a right basis for $N \times \mathcal{G}/N$.

Next, if $\pi_i(a\psi_0) \in \tilde{N}$, then

$$\begin{aligned}
E(\lambda^{(i)} \cdot a\psi_0 \cdot \lambda^{(j)b}) &= (\lambda^{(i)} \cdot a\psi_0)_j \text{ (by Lemma 10)} \\
&= (\lambda^{(i)} \theta(a))_j = \theta_k^i(1) \theta_j^k(a) = \theta_j^i(a);
\end{aligned}$$

this shows that θ_λ (the θ of $\mathcal{G}_\lambda$) may be identified with θ . This shows in particular that $[N \times \mathcal{G}:N] = (\text{Tr } \theta_\lambda(1))^{-1} = (\text{Tr } \theta(1))^{-1} = \tau^{-1}$.

To complete the proof, we now verify that also $\psi_{0\lambda} = \psi_0$ and $\psi_{1\lambda} = \psi$, so that, in fact, $\mathcal{G}_\lambda = \mathcal{G}$.

$E(\lambda^{(i)b}) = E(\psi_0 \cdot \lambda^{(i)b}) = \psi_i$ (by Lemma 10) and

$$\begin{aligned}
(\psi_{1\lambda}^*)_{k}^{ij} &= \tau^{1/2} E(\lambda^{(i)} \cdot \lambda^{(j)} \cdot \lambda^{(k)b}) \\
&= \tau^{1/2} (\lambda^{(i)} \cdot \lambda^{(j)})_k \text{ (by Lemma 10)} \\
&= \lambda_p^{(i)} \theta_q^p(\lambda_r^{(j)}) \psi_k^{qr} \\
&= \theta_p^i(1) \theta_q^p(\theta_j^r(1)) \psi_k^{qr} \\
&= \theta_{qr}^{(2)ij}(1) \psi_k^{qr} \\
&= \psi_k^{ij} \text{ (by Prop. 3.11 (iii))},
\end{aligned}$$

and the proof is complete. □

We digress now to make a few observations on the duality theory in $\mathcal{G}$. The 'duality theorem' is the following result (cf. [5]): if $(M, N) \in \mathcal{G}$ and if $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \dots$ is the tower of the basic construction, then

$$(M_2, M_1) \cong (\text{Mat}_{[M:N]}(M), \text{Mat}_{[M:N]}(N)),$$

where by $\text{Mat}_r(N)$ (r a positive real number) we mean $Q(\text{Mat}_{I \times I}(N))Q$, where I is a finite set and Q is a projection in $\text{Mat}_{I \times I}(N)$ satisfying $\text{Tr } Q = r$.

In view of Proposition 3.7(i), it is natural, then, to let $\mathcal{G}_\lambda$ denote the 'dual covariant system' of $\mathcal{G}_\lambda$, where, if λ is a right basis for M/N , we let $\hat{\lambda} = \tau^{-1/2} e_0 \lambda$. This leads naturally to the following definition.

DEFINITION 19

If $\mathcal{G} = (I, N, \theta, \psi_0, \psi_1)$ is a covariant system, the dual of $\mathcal{G}$ is the covariant system $\hat{\mathcal{G}} = (I, N \times \mathcal{G}, \hat{\theta}, \hat{\psi}_0, \hat{\psi}_1)$ defined by

$$\hat{\theta}_j^i(\pi_i(\xi)) = \tau^{-1/2} \theta_m^i(\xi_m) \psi_j^{mn}, \quad \xi \in \mathcal{U}$$

$$\hat{\psi}_i = \tau^{1/2} \lambda_i^*, \text{ where } \lambda_i = \pi_i(\lambda^{(i)}) \text{ (cf. Lemma 10)}$$

and $\hat{\psi}_{ik}^i = \theta_k^i(1) \psi_i$. □

$(N \times \mathcal{G}) \times \mathcal{G} \cong \text{Mat}_{r-1}(N)$; this statement is more similar, in form, to Takesaki's duality theorem.

In view of the known (and fairly easily established) fact $(R, R^G) \cong (R \times G, R)$ for outer actions of finite groups on the hyperfinite II_1 factor R , we conclude this minor digression into duality theory with an identification of the covariant system associated with (N, N^G) , at least in the case of a specific model of an outer G -action on the hyperfinite II_1 factor.

Example 20. Let G be any finite group, and let $\Omega = G^{\mathbb{Z}^+}$ be equipped with the usual Borel structure and Haar measure. Then Ω admits a natural measure preserving G action thus: if $s \in G$, $\alpha = (\alpha_0, \alpha_1, \dots) \in \Omega$, then $s \cdot \alpha = (s\alpha_0, s\alpha_1, \dots)$. Let R denote the 'tail equivalence relation': $(\alpha, \beta) \in R$ iff $\alpha_j = \beta_j$ for all but finitely many j . Let N denote the von Neumann algebra of bounded left multiplication operators on $L^2(R)$ (the measure on R coming from Haar measure on Ω and counting measure on equivalence classes). Then N is the hyperfinite II_1 factor; if $x \in N$, we write $x(\alpha, \beta)$ for $(x\xi)(\alpha, \beta)$ where ξ is the indicator function of the diagonal. It is well-known and easily established that the equation $(\zeta_s x)(\alpha, \beta) = x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$ defines an outer action of G on N ; thus $x \in N^G$ iff $x(\alpha, \beta) = x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$ for almost every (α, β) , for each s in G .

Let us denote by $\hat{G}$ a set consisting of exactly one member from each equivalence class of irreducible unitary representations of G . If $\pi \in \hat{G}$, write $\mathcal{H}_\pi$ for the Hilbert space on which π represents G , and let $d_\pi = \dim \mathcal{H}_\pi$. Further, fix an orthonormal basis $\{\xi_i : 1 \leq i \leq d_\pi\}$ for $\mathcal{H}_\pi$. Now, define $I = \{(\pi, i, j) : \pi \in \hat{G}, 1 \leq i, j \leq d_\pi\}$, and for each $(\pi, i, j) \in I$, define an element $\lambda_{(\pi, i, j)}$ of N by

$$\lambda_{(\pi, i, j)}(\alpha, \beta) = \delta_{\alpha\beta} \langle \pi(\alpha_0^{-1})\xi_i, \xi_j \rangle \sqrt{d_\pi}.$$

In order to verify that $\lambda = ((\lambda_{(\pi, i, j)})) \in \text{Mat}_{I \times I}(N)$ is a right basis for N/N^G , note first that the conditional expectation of N onto N^G is implemented by the projection e_0 defined on $L^2(R)$ by $(e_0 \eta)(\alpha, \beta) = 1/|G| \sum_{s \in G} \eta(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$; so, if $(\pi, i, j) \in I$ and $x \in N$, we have

$$(\lambda_{(\pi, i, j)}^* e_0 \lambda_{(\pi, i, j)} x)(\alpha, \beta) = \frac{d_\pi}{|G|} \langle \pi(\alpha_0)\xi_j, \xi_i \rangle \sum_{s \in G} \langle \pi(\alpha_0^{-1}s)\xi_i, \xi_j \rangle x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta);$$

it follows that

$$\begin{aligned} & \left(\sum_{(\pi, i, j) \in I} \lambda_{(\pi, i, j)}^* e_0 \lambda_{(\pi, i, j)} x \right)(\alpha, \beta) \\ &= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{(\pi, i, j)} d_\pi \langle \pi(s)\xi_i, \pi(\alpha_0)\xi_j \rangle \langle \pi(\alpha_0)\xi_j, \xi_i \rangle \\ &= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{\pi, i} d_\pi \langle \pi(s)\xi_i, \xi_i \rangle \\ &= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{\pi} d_\pi \chi_\pi(s) = x(\alpha, \beta), \end{aligned}$$

where we have written χ_π for the character of π , used the fact that $\sum d_\pi \chi_\pi$ is the character χ_r associated with the regular representation, and the fact that

$$\chi_r(s) = \begin{cases} |G|, & \text{if } s = \text{identity of } G \\ 0, & \text{otherwise} \end{cases}$$

Hence $\lambda = ((\lambda_{(\pi, i, j)}))$ is a right basis for N/N^G . A fairly straightforward calculation (which makes use of the orthogonality relations for the matrix entries of the irreducible representations) shows that the constituents of the covariant system $\mathcal{G}_\lambda$ are given as follows:

$$(\theta_{(\pi, i, j)}^{(\pi, i, j)})(x)(\alpha, \beta) = \delta_{\pi\pi'} \delta_{ii'} \langle \pi(\alpha_0^{-1} \beta_0) \xi_{j'}, \xi_j \rangle x(\alpha, \beta), \quad x \in N^G,$$

$$\psi_{(\pi, i, j)} = \begin{cases} 1, & \text{if } \pi \text{ is the trivial representation,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} & (\psi_{(\pi_1, i_1, j_1), (\pi_2, i_2, j_2)}^{(\pi, i, j)})(\alpha, \beta) \\ &= \delta_{\alpha\beta} \left(\frac{d_\pi d_{\pi_1} d_{\pi_2}}{|G|} \right)^{1/2} \langle \varphi_{(\pi, \pi(\alpha_0^{-1} \beta_0) \xi_{j_1}, \xi_{j_2})}, \varphi_{(\pi_1 \otimes \pi_2, (\pi_1 \otimes \pi_2)(\alpha_0^{-1} \beta_0) \xi_{i_1} \otimes \xi_{i_2}, \xi_{j_1} \otimes \xi_{j_2})} \rangle \end{aligned}$$

where we have used the notation $\langle \varphi_1, \varphi_2 \rangle = 1/|G| \sum_{s \in G} \varphi_1(s) \overline{\varphi_2(s)}$ for functions on G and $\varphi_{(\pi, \xi, \eta)}(s) = \langle \pi(s) \xi, \eta \rangle$. (Note, in particular, that the matrix ψ_1 carries the Clebsch-Gordon data.)

More generally, if H is a subgroup of G (where G is still assumed finite), and if N is as above, a right basis for N^H/N^G may be constructed as follows: let $\hat{G}^H$ be a set consisting of one irreducible (unitary) representation π of G from each unitary equivalence class with the property that $\pi|_H$ contains the trivial representation of H ; thus if $\mathcal{H}_\pi$ is the representation space of π , then $V_\pi = \{\xi \in \mathcal{H}_\pi : \pi(H)\xi = \xi\} \neq \emptyset$. For each π in $\hat{G}^H$, fix orthonormal bases $\{\xi_j : 1 \leq j \leq d_\pi\}$ and $\{\eta_i : 1 \leq i \leq d_\pi^H\}$ for $\mathcal{H}_\pi$ and V_π respectively. Finally, let

$$I = \{(\pi, i, j) : \pi \in \hat{G}^H, 1 \leq i \leq d_\pi^H, 1 \leq j \leq d_\pi\},$$

and define $\lambda_{(\pi, i, j)} \in N^H$ by

$$\lambda_{(\pi, i, j)}(\alpha, \beta) = \delta_{\alpha\beta} \sqrt{d_\pi} \langle \pi(\alpha_0^{-1}) \eta_i, \xi_j \rangle.$$

It may then be verified, using the fact that

$$\sum_{\substack{\pi \in \hat{G}^H \\ 1 \leq i \leq d_\pi^H}} \frac{d_\pi}{[G:H]} \langle \pi(s) \eta_i, \eta_i \rangle = \begin{cases} 0, & \text{if } s \notin H \\ 1, & \text{if } s \in H, \end{cases}$$

that $\lambda = ((\lambda_{(\pi, i, j)}))$ is a right basis for N^H/N^G . The associated covariant system may be easily computed. $\square$

Finally, we would like to make the following remarks:

(1) If $(M, N) \in \mathcal{C}$ and if λ is a right basis for M/N , let $\lambda^{(n)}, \theta^{(n)}$ be defined as in §3. Let $\mathcal{U}_n = \{\xi \in \text{Mat}_{n \times n}(N) : \xi = \xi \theta^{(n)}(1)\}$. Since $\lambda^{(n)}$ is a right basis for M_{n-1} , it follows that there is a bijection between M_{n-1} and $\mathcal{U}_n$; we think of M_{n-1} as $\pi_1(\mathcal{U}_n)$ where $\mathcal{U}_n$ is the Hilbert algebra defined in the way $\mathcal{U}(=\mathcal{U}_1)$ was defined in this section. Let M_∞

$$\mathcal{H}_\infty = \{(\xi_n)_{n=1}^\infty : \xi_n \in \text{Mat}_{I_n \times \cdot}(L^2(N)), \xi_n = \xi_n \theta^{(n)}(1), \\ \xi_n = \xi_{n+1} \psi_n^* \forall n \text{ and } \sup_n \|\xi_n\|_2 < \infty\}$$

where the ψ_n are defined as in Proposition 9; one next defines

$$\mathcal{U}_\infty = \left\{ (\xi_n)_{n=1}^\infty \in \mathcal{H}_\infty : \xi_n \in \mathcal{U}_n \text{ for all } n \right. \\ \left. \text{and } \sup_n \|\pi_t(\xi_n)\| < \infty \right\}.$$

Then $\mathcal{H}_\infty$ admits a $\mathcal{U}_\infty$ action as follows: if $\xi \in \mathcal{U}_\infty$, $\eta \in \mathcal{H}_\infty$, we have

$$(\xi \cdot \eta)_n = \lim_{m \rightarrow \infty} (\pi_t(\xi_m) \eta_m) \psi_{m-1}^* \psi_{m-2}^* \cdots \psi_n^*;$$

if we denote the map $\eta \mapsto \xi \cdot \eta$ by $\pi_t(\xi)$, it can finally be shown that $M_\infty = \pi_t(\mathcal{U}_\infty)$.

(2) It should be noted that the definition of a covariant system clearly makes sense when N is any finite von Neumann algebra, and that a large proportion of the results continue to hold there. In fact the author can prove the following result: there exists a covariant system $\mathcal{G} = (I, N, \theta, \psi_0, \psi_1)$ with N a finite-dimensional von Neumann algebra if and only if there exists a rectangular matrix A with non-negative integral entries such that τ is the Perron-Frobenius eigenvalue of AA' . It is the author's belief that it should be possible to prove the 'if' part of the above assertion, with 'finite-dimensional von Neumann algebra' replaced by 'the hyperfinite II_1 factor', by using the model for AF-algebras discussed in [7] to explicitly write down the outer covariant system which does the job.

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Existence of solution for nonlinear Volterra integral equations

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Abstract. We prove an existence theorem for a class of nonlinear Volterra integral equations.

Keywords. Nonlinear Volterra integral equation; deviating arguments.

1. Introduction

The theory of nonlinear Volterra integral equations with deviating arguments and functional integral equations have been studied by many authors [1, 2, 5, 7]. Banaś[4] has proved an existence theorem for functional integral equation and Balachandran[1] has proved an existence theorem for a nonlinear Volterra integral equation with deviating argument. In this paper we shall derive a set of sufficient conditions for the existence of a solution of nonlinear Volterra integral equations with the kernel containing a continuous operator.

2. Basic assumptions

Let $p(t)$ be a given continuous function defined on the interval $[0, \infty)$ and taking real positive values. Denote by $C_p = C([0, \infty), p(t); R^n)$ the set of all continuous functions from $[0, \infty)$ into R^n such that

$$\sup \{ |x(t)| p(t) : t \geq 0 \} < \infty.$$

It has been proved [6] that C_p forms a real Banach space with regard to the norm

$$\|x\| = \sup \{ |x(t)| p(t) : t \geq 0 \}.$$

If $x \in C_p$ then we will denote $\omega^T(x, \varepsilon)$ the usual modulus of continuity of x on the interval $[0, T]$ i.e.,

$$\omega^T(x, \varepsilon) = \sup \{ |x(t) - x(s)| : |t - s| \leq \varepsilon, t, s \in [0, T] \}.$$

Our existence theorem is based on the following lemma.

Lemma 3. Let E be a bounded set in the space C_p . If all functions belonging to E are equicontinuous on each interval $[0, T]$ and $\lim_{T \rightarrow \infty} \sup \{ |x(t)| p(t) : t \geq T \} = 0$ uniformly

Consider the nonlinear Volterra integral equation of the form

$$x(t) = G(x)g(t, x(h_1(t)), x(h_2(t)), \dots, x(h_m(t))) + \int_0^t K(t, s, Tx(s)) ds \quad (1)$$

where x and K are n -vectors and G is a real-valued function and $T: R^n \rightarrow R^n$ is a continuous operator. Assume the following conditions:

(i). Let $\Delta = \{ (t, s): 0 \leq s \leq t < \infty \}$.

The kernel $K: \Delta \times R^n \rightarrow R^n$ is continuous and there exist continuous functions $m: \Delta \rightarrow [0, \infty)$, $a: [0, \infty) \rightarrow (0, \infty)$ $b: [0, \infty) \rightarrow [0, \infty)$ such that

$$|K(t, s, x)| \leq m(t, s) + a(t)b(s)|x|$$

for all $(t, s) \in \Delta$ and $x \in R^n$.

In order to formulate other assumptions let us define

$$L(t) = \int_0^t a(s)b(s) ds, \quad t \geq 0.$$

Take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t) \exp(ML(t) + t)]^{-1}$.

(ii) there exists a constant $A > 0$ such that for any $t \in [0, \infty)$ the following inequality holds

$$\int_0^t m(t, s) ds \leq Aa(t) \exp[ML(t)]$$

(iii) for $i = 1, 2, \dots, m$ the functions $h_i: [0, \infty) \rightarrow [0, \infty)$ are continuous, $h_i(t) \leq t$ for $t \geq 0$ and there exists a positive real number B_i such that $a(h_i(t)) \leq B_i a(t)$,

(iv) $G: C_p \rightarrow [0, \infty)$ is continuous and bounded.

Assume $|G(x)| \leq k_1$ where k_1 is a positive constant.

(v) the function $g: [0, \infty) \times R^m \rightarrow R^n$ is continuous and satisfies the condition

$$|g(t, x_1, \dots, x_m) - g(t, y_1, \dots, y_m)| \leq \sum_{i=1}^m \alpha_i(t) |x_i - y_i|,$$

where $\alpha_i(t)$ is continuous and such that

$$\alpha_i(t) \leq \exp \{ M[L(t) - L(h_i(t))] \} \text{ for } t \geq 0, \quad \text{for } i = 1, 2, \dots, m$$

and $|g(t, 0, \dots, 0)| \leq a(t) \exp(ML(t))$,

(vi) the function $T: R^n \rightarrow R^n$ is continuous and

$$|Tx(t)| \leq a(t) \int_0^t b(s) |x(s)| ds,$$

(vii) Assume that $k_1(1+B) + A + (1/M^2) < 1$ where $B = \sum_{i=1}^m B_i$.

3. Existence theorem

Theorem. Assume that the hypotheses (i) to (vii) hold; then eq. (1) has at least one solutions x in the space C_p such that $|x(t)| \leq a(t) \exp [ML(t)]$ for any $t \geq 0$.

Proof. Define a transformation F in the space C_p by

$$(Fx)(t) = G(x)g(t, x(h_1(t)), \dots, x(h_m(t))) + \int_0^t K(t, s, Tx(s)) ds. \quad (2)$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0, \infty)$. Define the set E in C_p by

$$E = \{x \in C_p : |x(t)| \leq a(t) \exp [ML(t)]\}.$$

Clearly E is nonempty, bounded, convex and closed in C_p . Now we prove that F maps the set E into itself. Take $x \in E$. Then from our assumptions we have

$$\begin{aligned} |(Fx)(t)| &\leq |G(x)| |g(t, x(h_1(t)), \dots, x(h_m(t)))| + \int_0^t |K(t, s, Tx(s))| ds \\ &\leq k_1 |g(t, x(h_1(t)), \dots, x(h_m(t))) - g(t, 0, \dots, 0)| \\ &\quad + k_1 |g(t, 0, \dots, 0)| + \int_0^t m(t, s) ds + a(t) \int_0^t b(s) |Tx(s)| ds \\ &\leq k_1 \sum_{i=1}^m \alpha_i(t) |x(h_i(t))| + k_1 a(t) \exp [ML(t)] + Aa(t) \exp [ML(t)] \\ &\quad + a(t) \int_0^t b(s) \left[a(s) \int_0^s b(u) |x(u)| du \right] ds \\ &\leq k_1 \sum_{i=1}^m \exp [M[L(t) - L(h_i(t))]] a(h_i(t)) \exp [ML(h_i(t))] \\ &\quad + k_1 a(t) \exp [ML(t)] + Aa(t) \exp [ML(t)] \\ &\quad + a(t) \int_0^t b(s) \left[a(s) \int_0^s b(u) a(u) \exp [ML(u)] du \right] ds \\ &\leq k_1 Ba(t) \exp [ML(t)] + k_1 a(t) \exp [ML(t)] + Aa(t) \exp [ML(t)] \\ &\quad + (1/M)a(t) \int_0^t b(s)a(s) \exp [ML(s)] ds \\ &\leq [k_1(1 + B) + A + (1/M^2)]a(t) \exp [ML(t)] \\ &< a(t) \exp [ML(t)] \end{aligned}$$

which proves that $FE \subset E$.

Now we want to prove that F is continuous on the set E . In order to do this take $F = F_1 + F_2$, where

$$(F_1 x)(t) = G(x)g(t, x(h_1(t)), \dots, x(h_m(t))),$$

$$(F_2 x)(t) = \int_0^t K(t, s, Tx(s)) ds.$$

We shall prove continuity of F_1, F_2 separately. Let us fix $\varepsilon > 0$ and take $x, y \in E$ such that $\|x - y\| \leq \varepsilon$. We have

$$\begin{aligned} \|F_1 x - F_1 y\| &\leq k_1 \sup \{ |g(t, x(h_1(t)), \dots, x(h_m(t))) \\ &\quad - g(t, y(h_1(t)), \dots, y(h_m(t)))| \cdot [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sup \{ |g(t, y(h_1(t)), \dots, y(h_m(t)))| \\ &\quad \times [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \}. \\ &\leq k_1 \sum_{i=1}^m \sup \{ |\alpha_i(t) x(h_i(t)) - y(h_i(t))| [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sup \{ |g(t, y(h_1(t)), \dots, y(h_m(t))) - g(t, 0, \dots, 0)| \\ &\quad \times [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sup \{ |g(t, 0, \dots, 0)| \\ &\quad \times [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \}. \\ &\leq k_1 \sum_{i=1}^m B_i \sup \{ |x(h_i(t)) - y(h_i(t))| [a(h_i(t))]^{-1} \\ &\quad \times \exp(M[L(t) - L(h_i(t))]) \exp(-ML(t) - t) : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sum_{i=1}^m \sup \{ \alpha_i(t) |y(h_i(t))| \\ &\quad \times [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sup \{ \exp(-t) : t \geq 0 \} \\ &\leq k_1 \sum_{i=1}^m B_i \sup \{ |x(h_i(t)) - y(h_i(t))| \\ &\quad \times [a(h_i(t)) \exp(ML(h_i(t)) + h_i(t))]^{-1} : t \geq 0 \} \\ &\quad + |G(x) - G(y)| \sum_{i=1}^m \sup \{ a(h_i(t)) \exp(ML(t)) \\ &\quad \times [a(t) \exp(ML(t) + t)]^{-1} : t \geq 0 \} + |G(x) - G(y)| \\ &\leq k_1 B \|x - y\| + (B + 1) |G(x) - G(y)|. \end{aligned}$$

this implies that F_1 is continuous in view of (iv).

Now we prove that F_2 is continuous on the set E . For this let us fix $\varepsilon > 0$ and $x, y \in E$ such that $\|x - y\| \leq \varepsilon$. Further, take an arbitrary fixed $T > 0$. In view of (i) the function $K(t, s, x)$ is uniformly continuous on

where $r(T) = \max \{a(s) \exp(ML(s)) : s \in [0, T]\}$. Thus, we have for $t \in [0, T]$

$$\begin{aligned} |(F_2 x)(t) - (F_2 y)(t)| &\leq \int_0^t |K(t, s, Tx(s)) - K(t, s, Ty(s))| ds \\ &\leq \beta(\varepsilon) \end{aligned} \quad (3)$$

where $\beta(\varepsilon)$ is some continuous function such that $\lim_{\varepsilon \rightarrow 0} \beta(\varepsilon) = 0$. Further, let us take $t \geq T$. Then we have

$$\begin{aligned} |(F_2 x)(t) - (F_2 y)(t)| &\leq |(F_2 x)(t)| + |(F_2 y)(t)| \\ &\leq 2a(t) \exp(ML(t)). \end{aligned}$$

Consequently

$$|(F_2 x)(t) - (F_2 y)(t)| p(t) \leq 2 \exp(-t).$$

Hence for sufficiently large T we have

$$|(F_2 x)(t) - (F_2 y)(t)| p(t) \leq \varepsilon. \quad (4)$$

By (3) and (4) we get that F_2 is continuous on the set E . Hence $F = F_1 + F_2$ is continuous on E .

Now we prove that FE is relatively compact. For every $x \in E$ we have $Fx \in E$ which gives $|(Fx)(t)| p(t) \leq \exp(-t)$. Hence $\lim_{T \rightarrow \infty} \sup \{|(Fx)(t)| p(t) : t \geq T\} = 0$ uniformly with respect to $x \in E$.

Furthermore, let us fix $\varepsilon > 0$, $T > 0$; $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Then for $x \in E$, we have

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| &\leq |G(x)| |g(t, x(h_1(t)), \dots, x(h_m(t))) \\ &\quad - g(s, x(h_1(s)), \dots, x(h_m(s)))| \\ &\quad + \left| \int_0^t K(t, u, Tx(u)) du - \int_0^s K(s, u, Tx(u)) du \right| \\ &\leq k_1 \omega^T(g, \varepsilon) + \left| \int_0^t K(t, u, Tx(u)) du - \int_0^s K(t, u, Tx(u)) du \right| \\ &\quad + \left| \int_0^s K(t, u, Tx(u)) du - \int_0^s K(s, u, Tx(u)) du \right| \\ &\leq k_1 \omega^T(g, \varepsilon) + \int_s^t |K(t, u, Tx(u))| du \\ &\quad + \int_0^s |K(t, u, Tx(u)) - K(s, u, Tx(u))| du \\ &\leq k_1 \omega^T(g, \varepsilon) + \varepsilon \max \{m(t, u) + a(t)b(u) | Tx(u)| : 0 \leq u \leq t \leq T\} \end{aligned}$$

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in E$ such that $(Fx)(t) = x(t)$. Hence the theorem.

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Effect of a viscous fluid flow past a spherical gas bubble on the growth of its radius

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Abstract. A nearly spherical gas bubble expands adiabatically in a viscous incompressible fluid flowing past it. The Rayleigh-Plesset formula for the growth of the bubble radius is modified due to the flow of the viscous fluid.

Keywords. Rayleigh-Plesset formula; gas bubble.

1. Introduction

Rayleigh ([4]) obtained the basic differential equation, for the growth of the radius $R(t)$ with time t , of a gas bubble that expands adiabatically in an incompressible and inviscid fluid. Plesset ([6]) modified Rayleigh's equation taking into account the viscosity of the fluid surrounding the gas bubble.

The modification of Rayleigh's formula for the growth of the radius $R(t)$ with time t , due to the flow of an inviscid, incompressible fluid past a nearly spherical gas bubble, has been recently obtained ([3]). In this note we obtain the modification of Rayleigh's formula taking into account the viscosity, as well as the flow, of a fluid past the expanding gas bubble. Our result is thus a modification of the Rayleigh-Plesset formula, which is obtained when one considers the flow of a viscous fluid past an expanding gas bubble.

Ceschia and Nabergoj [2] have considered the motion of a nearly spherical gas bubble in a viscous liquid using the Lagrangian procedure and obtained the Rayleigh-Plesset formula. Adopting the same procedure, we shall give a straightforward derivation of the modification of Rayleigh-Plesset formula due to the flow of the viscous fluid surrounding the gas bubble. In this straightforward derivation we neglect the deviation of the gas bubble shape from the spherical form, although, as in the inviscid case [3], in the case of the viscous flow past an expanding gas bubble also, the shape of a gas bubble always deviates from the spherical form. Such a deviation is however small, when the radius of the bubble is sufficiently small [1]. The straightforward derivation will thus give us accurate result for a gas bubble of sufficiently small radius, rising freely in a viscous fluid.

2. Analysis of the problem

radius of the bubble is R_0 . The fluid outside the bubble flows past it and has a uniform velocity U at a large distance from it.

As we consider the fluid flowing past the bubble to have a small viscosity, we take the flow to be irrotational, the fluid velocity being derived from a velocity potential function ϕ . Such an irrotational flow will give us a true description of the flow throughout the flow region except in the boundary layer ([2]) surrounding the bubble surface. The dissipation function F ([4] p. 581) is given by

$$F = \frac{\mu}{2} \int (\text{curl } \mathbf{u})^2 dv + \frac{\mu}{2} \int \nabla \mathbf{u}^2 \cdot d\mathbf{s} - \mu \int \mathbf{u} \times \text{curl } \mathbf{u} \cdot d\mathbf{s} \quad (1)$$

where the first term on the right side of (1) is a volume integral taken over the whole of the fluid region outside the bubble, and the other terms are surface integrals taken over the surface area A of the bubble surface. The direction of $d\mathbf{s}$ is along outward normal to the surface. As argued by Ceshia and Naborgoj, since the contribution of the transport of vorticity to the dissipation function F given by (1) can be shown to be small, the existence of the boundary layer in the present problem may be neglected ([1]).

The velocity potential function in the liquid flowing past the bubble, in a spherical polar system of coordinates in which the centre of the bubble is taken to be the origin and at rest, is given by

$$\phi = U \cos \theta (r + R^3/(2r^2)) + R^2 \dot{R}/r. \quad (2)$$

In (2), the term $U \cos \theta (r + R^3/(2r^2))$ represents the velocity potential of the fluid flow past a rigid sphere of radius R when the flow at infinity is of uniform velocity $-U$ along the axis of symmetry and the term $R^2 \dot{R}/r$ represents the contribution due to the change in $R(t)$ with time t . In our analysis we find it more convenient to use a coordinate system in which the fluid at infinity is at rest, so that the potential functions now takes the form [7]

$$\phi = U R^3 \cos \theta / (2r^2) + \dot{R} R^2 / r. \quad (3)$$

The kinetic energy of the fluid flow outside the bubble is given by

$$T = 1/2 \rho \int \mathbf{u}^2 dv \quad (4)$$

the integration in (4) is throughout the entire region outside the gas bubble. Since the fluid velocity $\mathbf{u}$ is given by

$$\mathbf{u} = -\nabla \phi. \quad (5)$$

we have from (4), in view of (3), the result

$$T = \pi \rho R^3 (2\dot{R}^2 + U^2/3). \quad (6)$$

T , as given by (6), will be taken to be the kinetic energy of the whole system, since, in calculating the kinetic energy we shall neglect the contribution of the motion of the gas and the vapour in the bubble.

The potential energy of the system is

$$V = - \int (p_i - p_o) dv + \sigma \int_A dA \quad (7)$$

where v is the volume of the gas bubble and $p_e(t)$ is the pressure exerted by the liquid at a large distance from the bubble, and p_i is pressure inside the bubble and σ is the surface tension acting on the bubble surface. As shown by Ceschia and Nabergoj [2], we can write

$$V = \frac{4\pi R^3}{3} \left\{ p_e(t) - \frac{p_{g0}}{(\gamma-1)} (R_0/R)^{3\gamma} + 3\sigma/R \right\}, \quad (8)$$

where we neglect the vapour pressure, and p_{g0} is the initial gas pressure in the bubble when its radius is R_0 . γ is the ratio of the two specific heats of the gas.

Since $\text{curl } \mathbf{u} = 0$ for the flow considered by us, the dissipation function F , in view of (1), (3) and (5), becomes

$$F = 2\pi\mu R(4\dot{R}^2 + 3U^2). \quad (9)$$

Defining the Lagrangian function L as

$$L = T - V, \quad (10)$$

and taking $R(t)$ and z , where z is the height of the centre of the bubble above a fixed horizontal plane, as the only independent coordinates in our problem, the Lagrangian equation becomes ([5])

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{R}} \right) - \frac{\partial L}{\partial R} = - \frac{\partial F}{\partial \dot{R}} \quad (11)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = - \frac{\partial F}{\partial \dot{z}}. \quad (12)$$

Equation (11), in view of (6), and (8)–(10), finally gives us

$$R\ddot{R} + 3\dot{R}^2/2 - U^2/4 + \frac{1}{\rho} \left\{ p_e - p_{g0}(R_0/R)^{3\gamma} + \frac{2\sigma}{R} + 4\mu\dot{R}/R \right\} = 0 \quad (13)$$

while (12) similarly gives

$$\frac{d}{dt} (UR^3) = -18\mu RU/\rho, \quad (14)$$

we have used the result $\dot{z} = U$ in deriving (13) and (14).

Equation (13) gives us the modification of Rayleigh's equation for the growth of $R(t)$, the radius of a spherical gas bubble expanding adiabatically in fluid, when we take into account the viscosity ($\mu \neq 0$) and the flow ($U \neq 0$) of the fluid surrounding

In the absence of viscosity ($\mu=0$), (14) shows that

$$UR^3 = k, \quad (15)$$

where k is a constant. Equation (15) has a simple interpretation. As the bubble rises in the liquid, and if the liquid displaced by the bubble were to move with the velocity of the bubble, momentum of the displaced bubble will be conserved.

Finally, in the absence of viscosity ($\mu=0$), (13), in view of (15), takes the form

$$\ddot{R} + 3\dot{R}^2/(2R) - \frac{k^2}{4R^7} + \frac{1}{\rho R} \left\{ p_e - p_{g0}(R_0/R)^{3\gamma} + \frac{2\sigma}{R} \right\} = 0. \quad (16)$$

Equation (16) shows, as it is apparent from the term $-k^2/4R^7$, the effect of fluid flow on the growth of bubble radius will be expected to be significant when the radius of the bubble is small.

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Affine Poisson structures

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Abstract. We establish a one-to-one correspondence between the set of all equivalence classes of affine Poisson structures (defined on the dual of a finite dimensional Lie algebra) and the set of all equivalence classes of central extensions of the Lie algebra by $\mathbb{R}$. We characterize all the affine Poisson structures defined on the duals of some lower dimensional Lie algebras. It is shown that under a certain condition every Poisson structure locally looks like an affine Poisson structure. As an application, we show the role played by affine Poisson structures in mechanics. Finally, we prove some involution theorems.

Keywords. Lie algebras; central extensions; Poisson structure; affine linearization; symplectic structures; momentum map; involution theorems.

Introduction

Affine Poisson structures (APS) emerge out of the foundations of Poisson geometry laid by Lie [13] [18], as natural objects of study. One may find the natural occurrence of these structures in the study of problems related to the equivariance of momentum maps of hamiltonian actions. Lichnerowicz and Madana have studied the riemannian and kaehlerian geometric studies of these structures [12] [14]. Some solutions of the classical Yang-Baxter equation are in one-to-one correspondence with the APS's whose constant parts are nondegenerate [5]. These are nice examples of bihamiltonian systems (in the sense of [15]). Thus the multifaced aspects of these structures make us feel that an extensive study of them is of sufficient interest and importance. In this paper we attempt to study some of the differential geometric features of these structures which are in relation with mechanics. There has been some progress in this area which we have not included in this paper (see [10] [12] [14]).

In this paper §1 contains a review of results required later. In §2, we observe that an affine Poisson structure is defined on the dual of some Lie algebra and study some basic properties of affine Poisson structures and their connection with the Chevalley-Eilenberg cohomology and the theory of central extensions of the Lie algebra. We compute all the APS's on some lower dimensional Lie algebras. In §3 it is shown that under a certain condition every Poisson structure on a manifold locally looks like an affine Poisson structure. This is precisely a solution to the affine linearization problem posed by the author and Viswanath [4]. In §4 we describe the leaves of the symplectic foliation of an APS and in §5 we show how APS's occur as obstruction of the equivariance of a momentum map. Section 6 gives a brief account of the process of obtaining APS's by the Marsden-Weinstein reduction. As these structures are

bihamiltonian systems one can expect some involution theorems. Section 7 contains two such.

As far as the theory of APS's is concerned we have made the paper self-contained.

1. Review of some results required

Let (M, P) be a Poisson manifold, i.e. $[P, P] = 0$ where $[,]$ denotes the Schouten bracket of any two contravariant tensors on M (for the definition of this bracket see [4]). We denote by $\{, \}_P$, the Poisson bracket defined by P (i.e., $\{f, g\}_P = P(df, dg)$ for $f, g \in C^\infty(M)$). Then $P^\#$, a $C^\infty(M)$ -linear map from $\chi^*(M)$, the set of all smooth 1-forms on M , into $\chi(M)$, the set of all smooth vector fields on M , is defined by $P^\#(\alpha)(\beta) = P(\alpha, \beta)$ for all $\alpha, \beta \in \chi^*(M)$. We write $P^\#(\alpha)$ as $\alpha^\#$ more frequently.

Lemma 1. If W, X, Y, Z are in $\chi(M)$ then the alternating Schouten bracket of $W \wedge X$ and $Y \wedge Z$ is given by

$$[W \wedge X, Y \wedge Z] = \frac{-2}{3}([W, Y] \wedge X \wedge Z - [W, Z] \wedge X \wedge Y \\ - [X, Y] \wedge W \wedge Z + [X, Z] \wedge W \wedge Y).$$

Note: For all purposes of this paper the expression in the above lemma may be taken as definition of the alternating Schouten bracket.

Recall that a derivation X of an algebra $\mathcal{A}$ is an endomorphism of $\mathcal{A}$ satisfying $X(fg) = fX(g) + X(f)g$ for all f, g in $\mathcal{A}$ and a p -derivation on $\mathcal{A}$ is a multilinear map on $\mathcal{A}$ into itself such that it is derivation in every argument when the others are held fixed.

Lemma 2. Let V be a finite dimensional real vector space. (a) A p -derivation of $C^\infty(V)$ is determined by its values on V^* . (b) Every p -multilinear map T on V^* into itself has a unique extension to a p -derivation $\tilde{T}$ of $C^\infty(V)$. (c) The map $T \rightarrow \tilde{T}$ preserves the alternating and symmetric Schouten brackets.

(Note that the Schouten bracket is defined for any two multilinear maps on V into itself also.)

Given any bilinear map S on V into $\mathbb{R}$, by S^b we mean the map $S^b: V \rightarrow V^*$ defined by $\langle S^b(v), w \rangle = S(v, w)$.

All manifolds, maps, tensors and functions are assumed to be smooth and all vector spaces and Lie algebras are finite dimensional over $\mathbb{R}$. All manifolds are finite dimensional.

Proofs of the above lemmas can be found in [2] [4]. In general any unexplained notation or definition can be had from [1] [4] [10].

2. Affine Poisson structures and their properties

Choose a coordinate system $x_1, \dots, x_n$ on $\mathbb{R}^n$. Consider $P_{ij} \in C^\infty(\mathbb{R}^n)$ for $i, j = 1, \dots, n$

$$P = \sum_{i,j=1}^n P_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}.$$

Then P is a Poisson structure on $\mathbb{R}^n$ if and only if

$$P_{ij} = -P_{ji} \text{ (i.e., } P \text{ is alternating),}$$

$$\sum_{i=1}^n \left(P_{li} \frac{\partial P_{jk}}{\partial x_i} + P_{lj} \frac{\partial P_{ki}}{\partial x_i} + P_{lk} \frac{\partial P_{ij}}{\partial x_i} \right) = 0 \text{ (i.e., } [P, P] = 0),$$

for every choice of $i, j, k = 1, \dots, n$.

Lie [13], Sudarshan and Mukunda [17] and Lichnerowicz [11] have proved the above result.

Let us look at some important choices of P_{ij} 's.

First choice: If P_{ij} are constants satisfying $P(1)$ then $P(2)$ is automatically satisfied and give a Poisson structure. Let us call this *constant Poisson structure* on $\mathbb{R}^n$.

Second choice: If P_{ij} are linear homogeneous smooth functions on $\mathbb{R}^n$ (that is, $P_{ij} = \sum_{l=1}^n a_{ij}^l x_l$ where $a_{ij}^l \in \mathbb{R}$) satisfying $P(1)$ (that is, if $a_{ij}^l = -a_{ji}^l$) then $P(2)$ is satisfied if and only if

$$\sum_{r=1}^n (a_{jk}^r a_{ri}^l + a_{ki}^r a_{rj}^l + a_{ij}^r a_{rk}^l) = 0$$

for all $i, j, k, l = 1, \dots, n$. In this case P is called *linear (or Lie) Poisson structure* (LP) on $\mathbb{R}^n$ [18].

Third choice: If P_{ij} are non-homogeneous, linear smooth functions on $\mathbb{R}^n$ (that is, $P_{ij} = \sum_{l=1}^n a_{ij}^l x_l + b_{ij}$ where $a_{ij}^l, b_{ij} \in \mathbb{R}$) satisfying $P(1)$ (that is, $a_{ij}^l = -a_{ji}^l$ and $b_{ij} = -b_{ji}$) then $P(2)$ is satisfied if and only if (1) and

$$\sum_{r=1}^n (b_{ri} a_{jk}^r + b_{rj} a_{ki}^r + b_{rk} a_{ij}^r) = 0$$

for all $i, j, k = 1, \dots, n$ are true. In this case P is called *affine Poisson structure* (AP) on $\mathbb{R}^n$ (see [2] [4] [12] [14]).

Remarks. (1) Suppose P is a constant Poisson structure of rank $2s$ on $\mathbb{R}^n$. Then by Proposition 3.1.2 of [1] there exists a coordinate system $y_1, \dots, y_n$ on $\mathbb{R}^n$ such that

$$P = \sum_{i=1}^s \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_{s+i}}.$$

(2) Suppose P is an LPS on $\mathbb{R}^n$. Then (1) shows that P defines a Lie algebra structure on $(\mathbb{R}^n)^*$ with the structure constants a_{ij}^l . This is the well-known Kirillov-Kostant-Souriau structure or the Lie-Poisson structure on the dual of the Lie algebra $(\mathbb{R}^n)^*$.

Note: For the rest of the paper we fix the following notation. $\mathcal{G}$ is an n -dimensional Lie algebra with the structure constants C_{ij}^l with respect to a basis $X_1, \dots, X_n$. The basis defines a coordinate system $x_1, \dots, x_n$ on $\mathcal{G}^*$. Then the LPS on $\mathcal{G}^*$ is given by

$$C = \sum_{i,j,l=1}^n C_{ij}^l x_l \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}.$$

We fix that

$$B = \sum_{i,j=1}^n b_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

with $b_{ij} \in \mathbb{R}$ satisfying $b_{ij} = -b_{ji}$. If $C + B$ is an APS then B is called the constant part of the APS.

DEFINITION 1.

An APS $C + B$ is called exact APS if $b_{ij} = \sum_{l=1}^n d_l C_{ij}^l$ for $d_l \in \mathbb{R}$.

(Note that for the above choice of b_{ij} 's $C + B$ is always an APS).

Theorem 1. An APS on $\mathcal{G}^*$ defines a new Lie algebra which is a central extension of $\mathcal{G}$ by $\mathbb{R}$. The converse is also true.

Proof. Let $C + B$ be an APS. Then

$$[C + B, C + B] = 0 \Leftrightarrow [C, C] + [B, C] + [C, B] + [B, B] = 0 \Leftrightarrow [B, C] = 0$$

since $[\cdot, \cdot]$ is $\mathbb{R}$ -bilinear, commutative when any one of the entries is a 2-tensor, $[C, C] = 0$, and $[B, B] = 0$. Now, by Lemma 2, it is easy to see that $[B, C] = 0$ if and only if

$$[B, C](X, Y, Z) = 0$$

for all $X, Y, Z \in \mathcal{G}$. But

$$[B, C](X, Y, Z) = B([X, Y], Z) + B([Y, Z], X) + B([Z, X], Y)$$

because $C(X, Y) = [X, Y]$ and $C(X, a) = 0$ for any constant a . This shows that $C + B$ is an APS if and only if B is a 2-cocycle of the Chevalley–Eilenberg cohomology (CEC) of the Lie algebra $\mathcal{G}$ with respect to the trivial representation of $\mathcal{G}$ on $\mathbb{R}$ [6]. We now treat $\mathbb{R}$ as an abelian Lie algebra of dimension one. We know that every 2-cocycle of CEC of $\mathcal{G}$ with values in an abelian Lie algebra $\mathcal{A}$ defines a central extension of $\mathcal{G}$ by $\mathcal{A}$. Hence B defines a central extension of $\mathcal{G}$ by $\mathbb{R}$.

Conversely, if $\mathcal{E}$ is a central extension of $\mathcal{G}$ by $\mathbb{R}$ then it defines a 2-cocycle B' of CEC and hence $C + B'$ is an APS on $\mathcal{G}^*$.

Remark. Let us denote by $\mathcal{E}$, the central extension of $\mathcal{G}$ by $\mathbb{R}$ due to $B \cdot \mathcal{E} = \mathcal{G} \oplus \mathbb{R}$. Then the Lie bracket on $\mathcal{E}$ is given by

$$[(X, a), (Y, b)]_{\mathcal{E}} = ([X, Y], B(X, Y)).$$

Then $\mathcal{E}$ has dimension equal to $\dim \mathcal{G} + 1$. Moreover one can treat $X_1, \dots, X_n, 1$ as a basis of $\mathcal{E}$ and then the structure constants E_{ij}^l of $\mathcal{E}$ with respect to the above basis are given by

$$E_{ij}^l = C_{ij}^l, E_{ij}^{l+1} = b_{ij}, E_{n+1, j}^l = 0$$

for $i, j, l = 1, \dots, n$.

Note: From nowonwards a 2-cocycle B on $\mathcal{G}$ means a CEC 2-cocycle of $\mathcal{G}$ with respect to the trivial representation of $\mathcal{G}$ on $\mathbb{R}$ unless otherwise stated. One can think of B^* as a 1-cocycle of CEC with respect to the coadjoint representation of $\mathcal{G}$.

Lemma 3. *An exact APS on $\mathcal{G}^*$ is given by an exact 2-cocycle and vice versa.*

Proof. Let B be an exact 2-cocycle. There exists a 1-cochain D such that $B = \delta D$ where δ is the CEC operator [6]. Suppose $D = \sum_{i=1}^n c_i \alpha_i$ where $c_i \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_n$ is the dual basis of $X_1, \dots, X_n$. We have

$$b_{ij} = B(X_i, X_j) = -D([X_i, X_j]) = -\sum_{l=1}^n c_l C_{ij}^l.$$

Since $\delta B = [C, B] = 0$, B gives an exact APS. Conversely, if $C + B$ is an exact APS with $b_{ij} = \sum_{l=1}^n d_l C_{ij}^l$ then take $D = -(\sum_{l=1}^n d_l \alpha_l)$. Then δD is the exact 2-cocycle required.

COROLLARY 2.

If $\mathcal{G}$ is semisimple then every APS on $\mathcal{G}^*$ is exact.

Proof. If $\mathcal{G}$ is semisimple then the second cohomology group is zero. Hence every 2-cocycle is exact.

Theorem 2. *Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie algebras and let $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ be a vector space isomorphism. Suppose $C + B$ is an APS on $\mathcal{G}^*$ and λ is the inverse of the adjoint map of φ . Then $\lambda_*(C + B)$ is an APS on $\mathcal{H}^*$ if and only if φ is a Lie algebra isomorphism.*

Proof. Easy.

DEFINITION 2.

Two APS's on $\mathcal{G}^*$ are said to be the same if there exists a Poisson isomorphism from $\mathcal{G}^*$ onto itself which carries one of the two APS's to the other.

Some important remarks. (1) Let $c_i \in \mathbb{R}$. Then $\{y_i = x_i + c_i\}_{i=1, \dots, n}$ is another coordinate system on $\mathcal{G}^*$ and the affine Poisson brackets $\{x_i, x_j\}_{C+B}$ and $\{y_i, y_j\}_{C+B}$ are equal and hence they define the same APS. Therefore

$$\{x_i, x_j\}_{C+B} = \sum_l C_{ij}^l x_l + b_{ij} = \{y_i, y_j\}_{C+B} = \sum_l C_{ij}^l y_l + b'_{ij} = \{y_i, y_j\}_{C+B'}$$

where B' is the cocycle with coefficients b'_{ij} given by

$$b'_{ij} = b_{ij} - \sum_l C_{ij}^l c_l.$$

Now, by Lemma 3, we can conclude that 2-cocycles which are cohomologous give rise to the same APS.

(2) The dimension of the space of different (up to isomorphism) APS's on $\mathcal{G}^*$ is equal to the dimension of $H^2(\mathcal{G}, \mathbb{R})$. Note that this number is equal to the dimension of $H^1(\mathcal{G}, \mathcal{G}^*)$, the first CEC group with respect to the coadjoint representation of $\mathcal{G}$.

(3) If $\mathcal{G}$ is semisimple then one can choose c_i 's above in such a way that all b_{ij} 's vanish.

Examples. By Theorem 1 we know that if we want APS's then we have to look for 2-cocycles. For the present, let us get satisfied by determining all APS's on the duals of some lower dimensional Lie algebras using Jacobson's classification [9]. Theorem 2 shows that it is enough if we determine all APS's for Lie algebras up to isomorphism. We first note that for any abelian Lie algebra the LPS is zero and hence any APS is a constant Poisson structure.

(a) Let $\mathcal{G}$ be a nonabelian, 2-dimensional Lie algebra. Then every bilinear skew symmetric map is a 2-cocycle.

(b) Let $\dim \mathcal{G} = 3$ and let $\mathcal{G}$ be nonabelian.

Case (1). If $\dim [\mathcal{G}, \mathcal{G}] = 1$ and $[\mathcal{G}, \mathcal{G}] \subset Z(\mathcal{G})$ then $\mathcal{G}$ has a basis X, Y, Z with Lie bracket relations $[X, Y] = 0$, $[Y, Z] = X$, $[X, Z] = 0$. Then every skew symmetric bilinear map is a 2-cocycle.

Case (2). If $\dim [\mathcal{G}, \mathcal{G}] = 1$, $[\mathcal{G}, \mathcal{G}] \not\subset Z(\mathcal{G})$ then $\mathcal{G}$ has a basis X, Y, Z with Lie bracket relations $[X, Y] = X$, $[Y, Z] = [X, Z] = 0$. If

$$A = a \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + b \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + c \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$$

for $a, b, c \in \mathbb{R}$ then A gives a 2-cocycle if and only if $c = 0$.

Case (3). If $\dim [\mathcal{G}, \mathcal{G}] = 2$ then $\mathcal{G}$ has a basis X, Y, Z with bracket relations $[X, Z] = \alpha X + \beta Y$, $[Y, Z] = \gamma X + \theta Y$, $[X, Y] = 0$ where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \theta \end{pmatrix}$$

is a nonsingular matrix with real entries. If A is as above then A gives a 2-cocycle if and only if $a(\alpha + \theta) = 0$.

Case (4). If $\dim [\mathcal{G}, \mathcal{G}] = 3$ then $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$. Then there exists a basis X, Y, Z such that the bracket relations are given either by

$$[X, Y] = Z, [Y, Z] = X, [X, Z] = -Y$$

or by

$$[X, Y] = Z, [Y, Z] = -X, [X, Z] = -Y.$$

Then, if A is as above then A always gives a 2-cocycle. One can use Lemma 1 to prove the above observations.

3. Affine linearization problem

In view of the splitting theorem of Weinstein [18] the local study of Poisson manifold can be reduced to the case where the rank of the Poisson manifold (by definition the rank of a Poisson manifold is the rank of the Poisson structure) at a point is zero. Weinstein has proved that a Poisson structure near any such point can be approximated by a linear one and then posed the linearization problem [18]. Later Bhaskara and Viswanath [2] [4] have posed the affine linearization problem as follows. Let (M, P) be a Poisson manifold and let $y^0 \in M$. What are the conditions under which P is locally isomorphic to an APS at y^0 ? In this section we answer this question. We assume that M can be of variable rank. By the splitting theorem there exists a neighbourhood U of y^0 and an isomorphism $\phi = \phi_S \times \phi_N$ from U onto a product $S \times N$ such that S is symplectic and the rank of N at $\phi_N(y^0)$ is zero.

Theorem 3. P is locally isomorphic to an APS if and only if N is linearizable.

Proof. For our purpose, in view of the splitting theorem, it is enough to consider P to be a Poisson structure on $\mathbb{R}^{2r+s}$ in the following form

$$P = \sum_{i=1}^r \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+r}} + \sum_{i < j, i, j=1}^s Q_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

and $y^0 = 0$, $N = \{y \in \mathbb{R}^{2r+s} | x_i = 0 \text{ for } i = 1, \dots, 2r\}$ where $x_1, \dots, x_{2r}, z_1, \dots, z_s$ is a coordinate system on $\mathbb{R}^{2r+s}$ and $Q_{ij} = \{z_i, z_j\}_P$ satisfying $Q_{ij}(0) = 0$. Write $n = 2r + s$. Suppose N is a linearizable Poisson submanifold. Then there exist a Lie algebra $\mathcal{H}$ with a basis $X_{2r+1}, \dots, X_{2r+s}$ and the structure constants $C_{2r+i, 2r+j}^{2r+l}$, $i, j, l = 1, \dots, s$, and a Poisson isomorphism $\psi': (z_1, \dots, z_s) \rightarrow (x_{2r+1}, \dots, x_{2r+s})$ where $\{x_{2r+k}\}_{k=1}^s$ is the coordinate system on $\mathcal{H}^*$ defined by $\{X_{2r+k}\}_{k=1}^s$ such that

$$\psi'_* \left(\sum_{i < j, i, j=1}^s Q_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \right) = \sum_{i < j, i, j=1}^s C_{2r+i, 2r+j}^{2r+l} x_{2r+l} \frac{\partial}{\partial x_{2r+i}} \wedge \frac{\partial}{\partial x_{2r+j}}.$$

Write $\mathcal{G} = (\mathbb{R}^{2r})^* \times \mathcal{H}$. Then the map

$$\psi: (x_1, \dots, x_{2r}, z_1, \dots, z_s) \rightarrow (x_1, \dots, x_{2r}, x_{2r+1}, \dots, x_{2r+s})$$

is a diffeomorphism from $\mathbb{R}^{2r+s}$ onto $\mathcal{G}^*$ and

$$\psi_*(P) = \sum_{i=1}^r \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+r}} + \sum_{i < j, i, j=1}^s C_{2r+i, 2r+j}^{2r+l} x_{2r+l} \frac{\partial}{\partial x_{2r+i}} \wedge \frac{\partial}{\partial x_{2r+j}}. \quad (4)$$

It is easy to show that the right hand side of (4) is a Poisson structure (using Lemma 1). Therefore ψ is a Poisson isomorphism. Now treat $(\mathbb{R}^{2r})^*$ as an abelian Lie algebra and give the product Lie algebra structure to $(\mathbb{R}^{2r})^* \times \mathcal{H}$. The structure constants of the new Lie algebra $\mathcal{G}$ are $C_{2r+i, 2r+j}^{2r+l}$ for $i, j, l = 1, \dots, s$ with all the remaining ones equal to zero. Therefore the right hand side of (4) is an APS on $\mathcal{G}^*$ with the affine

part given by

$$(b_{ij})_{(2r+s) \times (2r+s)} = \begin{pmatrix} 0 & I_{r \times r} & 0 \\ -I_{r \times r} & 0 & 0 \\ 0 & 0 & 0_{s \times s} \end{pmatrix}.$$

Conversely, suppose P is isomorphic to an APS on $\mathcal{G}^*$, the dual of an n -dimensional Lie algebra. Let $t_1, \dots, t_n$ be a coordinate system on $\mathbb{R}^n$ and let the rank of P at 0 be $2r$ so that P is of the form

$$P = \sum_{i=1}^r \frac{\partial}{\partial t_i} \wedge \frac{\partial}{\partial t_{i+r}} + \sum_{i < j, i, j=2r+1}^{2r+s} P_{ij} \frac{\partial}{\partial t_i} \wedge \frac{\partial}{\partial t_j} \quad (5)$$

where $n - s = 2r$. Then there exists a Poisson isomorphism $\varphi: \mathbb{R}^n \rightarrow \mathcal{G}^*$ such that

$$T\varphi\left(\frac{\partial}{\partial t_i}\right) = \frac{\partial}{\partial x_i} \quad i = 1, \dots, n \text{ and}$$

$$\varphi_*(P) = \sum_{i < j, i, j=1}^n (C_{ij}^l x_l + b_{ij}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \quad (6)$$

By (5) and (6) we conclude that $C_{ij}^l = 0$ for $i, j = 1, \dots, 2r$ and $l = 1, \dots, 2r + s$ and

$$(b_{ij})_{n \times n} = \begin{pmatrix} 0 & I_{r \times r} & 0 \\ -I_{r \times r} & 0 & 0 \\ 0 & 0 & (b_{lk})_{s \times s} \end{pmatrix}$$

where $l, k = 2r + 1, \dots, 2r + s$. Moreover,

$$P_{ij} \circ \varphi^{-1} = \sum_{l=2r+1}^{2r+s} C_{ij}^l x_l + b_{ij}$$

for $i, j = 2r + 1, \dots, 2r + s$. The term on the right hand side of the above equation satisfies $P(1)$ and $P(2)$. Therefore it defines an APS on the dual of the Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$ determined by the part $X_{2r+1}, \dots, X_{2r+s}$ of the basis of $\mathcal{G}$ and the structure constants $\{C_{2r+i, 2r+j}^l\}_{i, j, l=1}^s$. Let $\varphi(0) = (x_1^0, \dots, x_{2r}^0, x_{2r+1}^0, \dots, x_{2r+s}^0)$. Then

$$b_{ij} = - \sum_{l=2r+1}^{2r+s} C_{ij}^l x_l^0$$

for $i, j = 2r + 1, \dots, 2r + s$ because $P_{ij}(0) = 0$ (by the splitting theorem). So the above APS is an exact APS. Consider the Poisson isomorphism from $\mathcal{H}^*$ onto $\mathcal{H}^*$ given by

$$(x_{2r+1}, \dots, x_{2r+s}) \rightarrow (x_{2r+1} - x_{2r+1}^0, \dots, x_{2r+s} - x_{2r+s}^0).$$

Under this Poisson isomorphism the above exact APS goes to the LPS on $\mathcal{H}^*$. Thus the Poisson manifold $N = \{t \in \mathbb{R}^n: t_i = 0 \text{ for } i = 1, \dots, 2r\}$ is linearizable.

Keen attention has been paid to the linearization problem and a substantial amount of work has been done [7] [8] [19]. In view of these results, Theorem 3 seems to be a satisfactory answer to the affine linearization problem.

4. Symplectic leaves of APS

Let M be any manifold. Let Φ be a left action of a Lie group G on M . Let $\mathcal{G}$ be the Lie algebra of G . The fundamental vector field associated with $X \in \mathcal{G}$ is a vector field X_M on M defined by

$$X_M(x) = \frac{d}{dt} \Phi(\exp(-tX), x)|_{t=0}$$

for $x \in M$. For any $X, Y \in \mathcal{G}$, $[X, Y]_M = [X_M, Y_M]$. Let $Ad(h) = T_h R_{h^{-1}} \circ T_e L_h$ for $h \in G$ (where e is the identity element of G). Then Ad^* and ad^* are defined by

$$\langle Ad^*(h)(\alpha), X \rangle = \langle \alpha, Ad(h^{-1})(X) \rangle$$

and

$$ad^*(X)(\alpha) = \frac{d}{dt} (Ad^*(\exp tX)(\alpha))|_{t=0}$$

for $X \in \mathcal{G}$, $\alpha \in \mathcal{G}^*$ respectively. Therefore $\langle ad^*(X)(\alpha), Y \rangle = -\langle \alpha, [X, Y] \rangle$ and $X_{\mathcal{G}^*}(\alpha) = -ad^*(X)(\alpha)$.

If $\alpha \in \mathcal{G}^*$ we identify $T_\alpha(\mathcal{G}^*)$ with $\mathcal{G}^*$ itself as follows: If $\beta \in \mathcal{G}^*$ then $\beta(f)(\alpha) = (d/dt)f(\alpha + t\beta)|_{t=0}$ for all $f \in C^\infty(\mathcal{G}^*)$. Consequently, vector fields on $\mathcal{G}^*$ may be identified with maps from $\mathcal{G}^*$ into $\mathcal{G}^*$.

For $f \in C^\infty(\mathcal{G}^*)$, $\alpha \in \mathcal{G}^*$, let $Df(\alpha)$ denote the Frechet derivative of f at α . Then $Df(\alpha)$ may be identified with an element of $\mathcal{G}$. Further,

$$\langle \beta, Df(\alpha) \rangle = \frac{d}{dt} f(\alpha + t\beta)|_{t=0}$$

for $\beta \in \mathcal{G}^*$. With the above notation one can write an expression for the Lie-Poisson bracket of $f, g \in C^\infty(\mathcal{G}^*)$ as

$$\{f, g\}_C(\alpha) = \langle \alpha, [Df(\alpha), Dg(\alpha)] \rangle.$$

Let $C + B$ be an APS then the affine Poisson brackets can be written as

$$\{f, g\}_{C+B}(\alpha) = \langle \alpha, [Df(\alpha), Dg(\alpha)] \rangle + B(Df(\alpha), Dg(\alpha)).$$

Therefore X_f , the hamiltonian vector field of $f \in C^\infty(\mathcal{G}^*)$ with respect to the above APS, is given by

$$X_f(\alpha) = ad^*(Df(\alpha))(\alpha) - B^p(Df(\alpha)).$$

By a standard theorem in Poisson geometry [4] [18], one can easily see that the symplectic leaves of the APS are precisely the integral submanifolds of the involutive distribution defined by $\{X_f | f \in C^\infty(\mathcal{G}^*)\}$.

Remark. Note that given a 2-cocycle A of $\mathcal{G}$ one can define two actions $aff_\pm$ of $\mathcal{G}$

on $\mathcal{G}^*$ by

$$\langle aff_+(X)(\xi), Y \rangle = \langle \xi, [X, Y] \rangle + A(X, Y)$$

and

$$\langle aff_-(X)(\xi), Y \rangle = \langle \xi, [X, Y] \rangle - A(X, Y)$$

for $X, Y \in \mathcal{G}, \xi \in \mathcal{G}^*$. Let us call them the affine actions of $\mathcal{G}$ on $\mathcal{G}^*$ corresponding to $\pm A$.

5. Momentum map and APS

Let (M, ω) be a symplectic manifold and let a Lie group G be acting on M by symplectic diffeomorphisms. Denote these morphisms by Φ_h for $h \in G$. Let $x \in M$. Suppose a momentum map J exists for the above action. Then one can define $\hat{J}: \mathcal{G} \rightarrow C^\infty(M)$ by $\hat{J}(X)(x) = \langle J(x), X \rangle$. Now there are two important questions [1] [10].

Q1) Is J an Ad^* -equivariant map? (i.e., $J(\Phi_h(x)) = Ad^*(h)(J(x))$).

Q2) Is $\hat{J}$ a Lie algebra homomorphism? (i.e., $\hat{J}([X, Y]) = \{\hat{J}(X), \hat{J}(Y)\}_\omega$). Here $\{, \}_\omega$ is the Poisson bracket due to the symplectic structure ω .)

Theorem 4. 1) $\sigma(h) = J(\Phi_h(x)) - Ad^*(h)(J(x))$ is independent of $x \in M$ and is a coadjoint cocycle of the cohomology of G . Moreover, $\sigma: G \rightarrow \mathcal{G}^*$ satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + Ad^*(h_1)(\sigma(h_2)) \quad (7)$$

for all $h_1, h_2 \in G$.

2) $A(X, Y) = \hat{J}([X, Y]) - \{\hat{J}(X), \hat{J}(Y)\}_\omega$ is a 2-cocycle of the CEC cohomology of $\mathcal{G}$.

3) $A^b = T_e \sigma$.

Proof. See [1] [10].

A cocycle σ of G satisfying (7) is called symplectic cocycle of G .

Remarks. 1) The momentum mapping $J: M \rightarrow \mathcal{G}^*$ is a Poisson morphism with respect to $\{, \}_\omega$ and $\{, \}_{C+A}$.

2) Every inequivariant momentum map defines an APS of $\mathcal{G}^*$.

Souriau [16] has proved the following.

Theorem 5. Let J be a momentum mapping for the symplectic action Φ with a symplectic cocycle σ . Then

(1) the map $\Psi: (h, \mu) \rightarrow Ad^*(h)(\mu) + \sigma(h)$ is an action of G on $\mathcal{G}^*$.

(2) J is equivariant with respect to the action of (1).

Remarks. 1) Given any symplectic cocycle σ (need not be the one defined by some J) of G , one can define an APS $C + A$ on $\mathcal{G}^*$ by taking $A(X, Y) = \langle (T_e \sigma)(X), Y \rangle$ and an action of G on $\mathcal{G}^*$ using the formula of Theorem 5 (1). Then it can be verified that Ψ_h is a Poisson morphism between $(\mathcal{G}^*, C + A)$ and itself.

2) Given any symplectic cocycle σ one can define two types of affine actions of G with respect to σ on $\mathcal{G}^*$ as follows.

$$Aff_+(h, \mu) = Ad^*(h)(\mu) + \sigma(h)$$

Proof. We know that A^p satisfies the following property.

$$\langle A^p(X), Y \rangle = -\langle X, A^p(Y) \rangle$$

for $X, Y \in \mathcal{G}$. Now, for any $X \in \mathcal{G}$, one can define a 1-form α_X on G as follows.

$$\langle \alpha_X(h), v \rangle = \langle A^p \circ T_h L_{h^{-1}}(v), \text{Ad}(h^{-1})X \rangle$$

for $h \in G, v \in T_h G$. Then there exists a unique map $\sigma: G \rightarrow \mathcal{G}^*$ such that $\sigma(e) = 0$ and $d\sigma_X(e) = \alpha_X$ for all $X \in \mathcal{G}$ where $\sigma_X: G \rightarrow \mathbb{R}$ is defined by $\sigma_X(h) = \langle \sigma(h), X \rangle$. Now it is easy to verify that σ is a symplectic cocycle and $T_e \sigma = A^p$.

Thus we have three ways of tackling Q1 and Q2.

- 1) By eliminating, if possible, the coefficients of $T_e \sigma$.
- 2) By defining a new action of G on M as given in Theorem 5 (in this case J will be $\text{Aff}_{\pm}$ -equivariant).
- 3) By taking action of Lie group corresponding to the central extension of $\mathcal{G}$ by $\mathbb{R}$ due to $T_e \sigma$.

As the main aim of all the above efforts is to have J as a Poisson morphism, one has the following choice also.

- 4) By taking the APS $C + T_e \sigma$ on $\mathcal{G}^*$ one can have J as a Poisson morphism.

Example. Any symplectic action of a semisimple Lie group G on (M, ω) leads to an equivariant momentum map J (if exists). Here we give an example in which J is not equivariant. This example is well-known to physicists.

Let $M = (\mathbb{R}^{2n}, q_1, \dots, q_n, p_1, \dots, p_n)$ with the canonical symplectic structure $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ and let $G = \mathbb{R}^{2n}$ act on M by translations T_h . Take the canonical basis $E^1, \dots, E^{2n}$ of $\mathcal{G}$, the Lie algebra of G . Then, for $i = 1, \dots, n$, we have

$$\begin{aligned} E_M^i(q_1, \dots, q_n, p_1, \dots, p_n) &= \frac{d}{dt} T(\exp(-tE^i), (q_1, \dots, q_n, p_1, \dots, p_n))|_{t=0} \\ &= \frac{d}{dt} T((0, \dots, 0, \overset{\text{ith-place}}{-t}, 0, \dots, 0), (q_1, \dots, q_n, p_1, \dots, p_n))|_{t=0} \\ &= -(0, \dots, 0, 1, 0, \dots, 0) \\ &= -\frac{\partial}{\partial q_i}(q_1, \dots, q_n, p_1, \dots, p_n). \end{aligned}$$

Similarly one can prove that $E_M^{i+n} = -\partial/\partial p_i$. We know that $dq_i^\# = \partial/\partial p_i$ and $dp_i^\# = -\partial/\partial q_i$. Hence $\hat{J}$ maps E^i to p_i and E^{i+n} to $-q_i$. Therefore, for $i, j = 1, \dots, n$, we have $\{\hat{J}(E^i), \hat{J}(E^{j+n})\}_\omega = -\delta_{ij}$ (here the delta is the Kronecker delta). On the other hand $\hat{J}([E^i, E^{j+n}]) = 0$ since G is abelian. That means that $\hat{J}$ is not a Lie algebra homomorphism and $b_{i,i+n} = -1$, $b_{i+n,i} = 1$ and all other b_{jk} 's are equal to zero. Since all the structure constants vanish it follows from (3) that $b_{i,i+n}$'s cannot be changed by any other choice of coordinates on $\mathbb{R}^{2n}$. Therefore to get $\hat{J}$ as a Lie algebra homomorphism, the group G is replaced by its central extension, the Heisenberg group.

Suppose σ is a symplectic two-cocycle on G and $A = T_e \sigma$. Let ω be the canonical symplectic structure on T^*G . Consider the APS $C + A$ on $\mathcal{G}^*$. A can be treated as a left-invariant closed 2-form on T^*G since it is a 2-cocycle. Let $\pi: T^*G \rightarrow G$ be the canonical projection. Then $\Omega_\sigma = \omega + \pi^* A$ is a symplectic structure on T^*G . Let $\hat{R}_h$ and $\hat{L}_h (h \in G)$ be the canonical liftings of the right and the left translations to T^*G respectively. Then one can define two left actions of G on T^*G by

$$\Phi_L(h, \xi) = \hat{L}_h(\xi)$$

and

$$\Phi_R(h, \xi) = \hat{R}_{h^{-1}}(\xi) + \hat{L}_{\pi(\xi)h^{-1}}(\sigma(h))$$

where $h \in G$ and $\xi \in T^*G$. The above two actions are free and hamiltonian. They admit momentum maps J_L and J_R given by

$$J_L(\xi) = -Ad_{\pi(\xi)}^*(\hat{L}_{\pi(\xi)^{-1}}(\xi))$$

$$J_R(\xi) = \hat{L}_{\pi(\xi)^{-1}}(\xi).$$

J_L and J_R are submersions, because the above actions are free. Now, one can obtain, by the Marsden-Weinstein reduction, the APS $C + A$. Moreover, the action of G on $\mathcal{G}^*$, for which J_L (or J_R) is equivariant is, Aff_- (or Aff_+) with respect to σ and J_R (or J_L) is a Poisson morphism from the symplectic manifold (T^*G, Ω_σ) into $(\mathcal{G}^*, C - A)$ (or $(\mathcal{G}^*, C + A)$).

All the details of the above facts are meticulously worked out in [10].

7. Involution theorems

I). Let (M, P) be a Poisson manifold. Two functions $f, g \in C^\infty(M)$ are said to be in *involution* if $\{f, g\}_P = 0$. A function $f \in C^\infty(M)$ is called *central function* with respect to P if $\{f, g\}_P = 0$ for all $g \in C^\infty(M)$. Let us denote by Z_P , the set of all central functions with respect to P .

Theorem 6. Let us suppose the following. ω, π, σ, J_R are as in the previous section, $B = T_e \sigma$, $(\mathcal{G}^*, C + B)$ is an affine Poisson manifold, U is an open subset of $\mathcal{G}^*$, $\Gamma: U \rightarrow T^*G$ is a section of J_R and $\Gamma(U)$ is coisotropic submanifold of $(T^*G, \omega + \pi^* B)$. Then if $f, g \in C^\infty(T^*G)$ are central functions with respect to $\omega + \pi^* B$ then $f \circ \Gamma, g \circ \Gamma$ are in involution.

Proof. Note that J_R is a surjective submersion and a Poisson morphism. Let $a \in U$ and $\Gamma(a) = b$. Denote by E and F respectively $T_a(\Gamma(U))$ and $\text{Ker}(T_b J_R)$. Then we have $T_b(T^*G) = E \oplus F$ and hence have $T_b^*(T^*G) = F^\circ \oplus E^\circ$ ($^\circ$ denotes the annihilator). Therefore $T_b^*(T^*G)$ can be identified with $E^* \oplus F^*$. Let pr_1 and pr_2 be the canonical projections onto E^* and F^* along F^* and E^* respectively. Then we have

$$\{f \circ \Gamma, g \circ \Gamma\}_{C+B}(a) = (\omega + \pi^* B)(pr_1(df(b))^\#, pr_1(dg(b))^\#)$$

because J_R is a Poisson morphism. Then

$$\begin{aligned} \{f \circ \Gamma, g \circ \Gamma\}_{C+B}(a) &= (\omega + \pi^* B)((df(b) - pr_2(df(b)))^\#, (dg(b) - pr_2(dg(b)))^\#) \\ &= (\omega + \pi^* B)((pr_2(df(b)))^\#, ((pr_2(dg(b)))^\#) = 0 \end{aligned}$$

because f, g are central functions with respect to $\omega + \pi^*B$, $\Gamma(U)$ is coisotropic submanifold and $(E^\circ)^\# = E^\perp$ ($\perp$ is taken with respect to $\omega + \pi^*B$).

II). If $C + B$ is an APS, by the fact that $[B, C] = 0$, we get that

$$\{Z_C, Z_C\}_B \subset Z_C$$

i.e., Z_C is a Poisson subalgebra of $(C^\infty(\mathcal{G}^*), \{, \}_B)$.

Let ∂_B be the Poisson cohomology operator [2] [3] [4] [11] on the space of contravariant tensors of $\mathcal{G}^*$ with respect to B . Let $Y_1, \dots, Y_n$ be a basis of $\mathcal{G}$ with the property that the matrix $(B(Y_i, Y_j))_{ij}$ is of the form

$$\begin{pmatrix} 0 & I_{r \times r} & 0 \\ -I_{r \times r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $2r$ is the rank of B .

Let D_{ij}^l be the structure constants of $\mathcal{G}$ with respect to the basis $Y_1, \dots, Y_n$ and let $y_1, \dots, y_n$ be the coordinate system defined on $\mathcal{G}^*$ by the above basis. Then $C = \partial_B T + D$ for some vector field T and 2-vector field D by the proposition of §3 in Chapter I of [11] (see also [4]). It is easy to see that

$$D = \sum_{i, j = 2r+1 \text{ to } n, l = 1 \text{ to } n, l < j} D_{ij}^l y_l \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

Now for any two functions $f, g \in Z_B$ we have

$$\{f, g\}_C = T(\{f, g\}_B) - \{T(f), g\}_B - \{f, T(g)\}_B + \{f, g\}_D = \{f, g\}_D.$$

Therefore $\{Z_B, Z_B\}_C = 0$ if and only if $D = 0$.

Thus we have proved the following.

Theorem 7. $\{Z_B, Z_B\}_C = 0$ if and only if C is ∂_B -exact.

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On the existence of a finite invariant measure

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Abstract. A necessary and sufficient condition is given for a Borel automorphism on a standard Borel space to admit an invariant probability measure.

Keywords. Borel space; Borel automorphism; compressible sets; invariant measures.

1. Introduction

The purpose of this paper is to seek conditions under which a Borel automorphism T on a Borel space $(X, \mathcal{B})$ admits a non-trivial finite T -invariant measure. For simplicity we will assume that $\mathcal{B}$ contains singleton sets. It is easy to see that if T admits periodic points, such a measure always exists. Hence we assume that $T^n x \neq x$ for all $n \in \mathbb{Z} - \{0\}$. Going back to classical ergodic theory, especially the work of Hopf [5] on the existence of a finite invariant measure equivalent to a given quasi-invariant measure, we are able to define the notion of incompressible set intrinsically (i.e., without reference to any measure) as follows:

An invariant set $A \in \mathcal{B}$ is said to be compressible if we can write A as a disjoint union of two sets C and D in $\mathcal{B}$ such that

$$\bigcup_{n=-\infty}^{\infty} T^n C = \bigcup_{n=-\infty}^{\infty} T^n D = A$$

and A is equivalent by countable decomposition to either C or D . An invariant set in $\mathcal{B}$ is said to be incompressible if it is not compressible.

If there is a finite T -invariant probability measure on $\mathcal{B}$ then X is incompressible. In the converse direction we are able to prove, under a mild condition which is satisfied whenever $\mathcal{B}$ is countably generated, that if X is incompressible then there exists a function

$$m: \mathcal{B} \times X \rightarrow [0, 1] \text{ such that}$$

- (i) $m(A, \cdot)$ is measurable for all A , $m(X, x) = 1$
- (ii) $m(A, x) = m(TA, x) = m(A, Tx) \pmod{\mathcal{H}}$
- (iii) $m(A, x) \equiv 0 \pmod{\mathcal{H}}$ if and only if $A \in \mathcal{H}$
- (iv) $m(\bigcup_{i=1}^{\infty} A_i, x) = \sum_{i=1}^{\infty} m(A_i, x) \pmod{\mathcal{H}}$ whenever $A_1, A_2, A_3, \dots$ are pairwise disjoint sets in $\mathcal{B}$.
- (v) If $E \in \mathcal{B}$ is an invariant set (i.e., $TE = E$) then for $x \in E$, $m(A \cap E, x) = m(A, x) \pmod{\mathcal{H}}$.

Here $\mathcal{H}$ is the σ -ideal of all sets $A \in \mathcal{B}$ such that $\bigcup_{n=-\infty}^{\infty} T^n A$ is compressible. $\mathcal{H}$ is called the Hopf ideal.

The classical theorem of Hopf can be recovered by integrating the function m with respect to the quasi-invariant measure in the statement of Hopf's theorem ([3], [5]). We mention, however, that the purpose here is not to give yet another proof of Hopf's theorem, but rather to isolate the descriptive content in Hopf's definition of incompressibility. When $(X, \mathcal{B})$ is a standard Borel space the function m can be so chosen that (i) – (iv) hold for every x (i.e., not modulo $\mathcal{H}$) and the measures $(m(\cdot, x))$ yield all the probability measures on $\mathcal{B}$ invariant and ergodic under T . The method of proof, which we give in sufficient detail, relies on elementary ideas of ergodic theory [3], [4] and the method of Murray and von-Neumann [6] wherein they obtain a measure on the class of projections in a factor. This paper may be viewed as a contribution to the theme of descriptive ergodic theory [1], [2], [7]–[12]. The paper considers the problem of the existence of a finite invariant measure formulated in [8]. The result here differs from the one in [7] in that the σ -ideal with respect to which incompressibility is defined is described intrinsically and no assumption is made about the ergodicity of T with respect to the σ -ideal.

2. Preliminaries and the main theorem

2.1 Let $(X, \mathcal{B})$ be a Borel space, i.e., a non-empty set X together with a σ -algebra $\mathcal{B}$ of subsets of X . Let $T: X \rightarrow X$ be a one-one onto map such that $T\mathcal{B} = T^{-1}\mathcal{B} = \mathcal{B}$. Such a map is called a Borel automorphism of $(X, \mathcal{B})$. We will assume that T has no periodic points. For any $A \subseteq X$ we write $sA = \bigcup_{n=-\infty}^{\infty} T^n A$ and call sA the saturation of A . It is the smallest T -invariant subset of X containing A . We write $s^+ A = \bigcup_{n=0}^{\infty} T^n A$.

2.2 A subset $A \subseteq X$ is said to be wandering if $T^n A \cap T^m A = \emptyset$ whenever $m \neq n$. The σ -ideal of $\mathcal{B}$ generated by all wandering sets in $\mathcal{B}$ is denoted by $\mathcal{W}$ and called the Shelah-Weiss ideal of T (see [8], [9], [11]).

2.3 Two sets $A, B \in \mathcal{B}$ are said to be equivalent by countable decomposition or descriptively isomorphic if (i) we can partition A into a countable number of pairwise disjoint sets $A_i \in \mathcal{B}$, $i \in \mathbb{N}$ (ii) we can partition B into a countable number of pairwise disjoint sets $B_i \in \mathcal{B}$, $i \in \mathbb{N}$ (iii) we can find integers n_i , $i \in \mathbb{N}$ such that for each i , $T^{n_i} A_i = B_i$. Here and in the sequel $\mathbb{N}$ denotes the set of natural numbers. If A and B are equivalent by countable decomposition then the map from A to B which is T^{n_i} on A_i , $i = 1, 2, 3, \dots$ is called a descriptive isomorphism between A and B . It can be shown that equivalence by countable decomposition is indeed an equivalence relation.

2.4 A set $A \in \mathcal{B}$ is said to be compressible if we can write A as a disjoint union of two sets $C, D \in \mathcal{B}$ such that $sA = sC = sD$ and A is equivalent to C by countable decomposition. A is said to be incompressible if A is not compressible. If X is compressible then T is called compressing Borel automorphism.

2.5 Orbit of a point is compressible and so is the saturation of any wandering set in $\mathcal{B}$. It is not true, however, that every compressible invariant set in $\mathcal{B}$ is the saturation of a wandering set in $\mathcal{B}$ except in special cases.

2.6 A finite non-empty set is not compressible nor is a set A compressible if the orbit of a point intersects A in a finite non-empty set. If there is a probability measure on $\mathcal{B}$ which is invariant under T , then no set of positive measure is compressible. In particular, in such a situation, X is incompressible. The purpose of this paper is to investigate the extent to which the incompressibility of X implies the existence of a finite T -invariant probability measure on $\mathcal{B}$. At least when $(X, \mathcal{B})$ is a standard Borel space we will show that incompressibility of X implies the existence of a T -invariant probability measure on $\mathcal{B}$.

2.7 Since finite non-empty sets are incompressible, it is clear that a subset of a compressible set need not be compressible.

2.8 A set $E \in \mathcal{B}$ is said to be T -invariant or simply invariant if $TE = E$. If $E \in \mathcal{B}$ is invariant and compressible and $F \in \mathcal{B}$ is an invariant subset of E , then F is compressible since a compression of E when restricted to F yields a compression of F . Countable pairwise disjoint union of compressible invariant sets in $\mathcal{B}$ is again compressible. Finally any countable union of compressible invariant sets in $\mathcal{B}$ is compressible since such a union can be expressed as a countable pairwise disjoint union of compressible invariant sets in $\mathcal{B}$.

2.9 Compressible sets in $\mathcal{B}$ do not form a σ -ideal but compressible invariant sets in $\mathcal{B}$ are closed under countable union and taking of invariant subsets in $\mathcal{B}$. Consequently the collection $\mathcal{H}$ of sets in $\mathcal{B}$ whose saturations are compressible form a σ -ideal in $\mathcal{B}$ which we call the Hopf ideal. We note that $\mathcal{W} \subseteq \mathcal{H}$ since the saturation of any set in $\mathcal{W}$ is the union of all the iterates under T of a suitable wandering set in $\mathcal{B}$.

2.10 A set A in $\mathcal{B}$ is said to be decomposable if we can write A as a disjoint union of two sets C and D in $\mathcal{B}$ such that $sA = sC = sD$.

2.11 Not every set in $\mathcal{B}$ is decomposable. For example a singleton set in $\mathcal{B}$ is not decomposable nor can a set in $\mathcal{B}$ be decomposable if it intersects an orbit in exactly one point. In the sequel we will need decomposability (mod $\mathcal{W}$). A set $A \in \mathcal{B}$ is said to be decomposable (mod $\mathcal{W}$) if there is a set $W \in \mathcal{W}$ such that $A \Delta W$ is decomposable. It can be shown that when $\mathcal{B}$ is countably generated every non-empty set in $\mathcal{B}$ is decomposable (mod $\mathcal{W}$) see [8].

2.12 Let $\mathfrak{n} \subseteq \mathcal{B}$ be a σ -ideal such that (i) $T\mathfrak{n} = T^{-1}\mathfrak{n} = \mathfrak{n}$ (ii) $\mathcal{W} \subseteq \mathfrak{n}$. We call such a system $(X, \mathcal{B}, \mathfrak{n}, T)$ a descriptive dynamical system. The quadruplets $(X, \mathcal{B}, \mathcal{W}, T)$ $(X, \mathcal{B}, \mathcal{H}, T)$ are examples of such systems. If μ is a σ -finite measure on $\mathcal{B}$ whose null sets in $\mathcal{B}$ are preserved under T and T is conservative, then $(X, \mathcal{B}, \mathfrak{n}_\mu, T)$ is such a system where $\mathfrak{n}_\mu = \mu$ null sets in $\mathcal{B}$. If two sets $A, B \in \mathcal{B}$ are such that $A \Delta B \in \mathfrak{n}$ then we say that A and B are equal (mod $\mathfrak{n}$) and write $A = B$ (mod $\mathfrak{n}$). It is well known and easy to prove that (mod $\mathcal{W}$) the sets sA and $s_+ A$ are equal and more generally the sets sA and $T^n s_+ A$ are equal (mod $\mathcal{W}$) (see [8] 5.4). Since $\mathcal{W} \subseteq \mathfrak{n}$ we also have $sA = T^n s_+ A$ (mod $\mathfrak{n}$) for all $A \in \mathcal{B}$ and $n \in \mathbb{Z}$.

2.13 Two sets $A, B \in \mathcal{B}$ are said to be equivalent by countable decomposition (mod $\mathfrak{n}$) or descriptively isomorphic (mod $\mathfrak{n}$) if we can find sets $N, M \in \mathfrak{n}$ such that $A - N$ and

We note that if $A \sim B \pmod{n}$ then $sA = sB \pmod{n}$. We say that A and B are strictly equivalent by countable decomposition if $A \sim B \pmod{\mathcal{W}}$. A set $A \in \mathcal{B}$ is said to be compressible $\pmod{n}$ if there is a set $N \in \mathcal{n}$ such that $A - N$ is compressible. In case A is compressible $\pmod{\mathcal{W}}$ we say that A is strictly compressible. For a T -invariant set in $\mathcal{B}$ compressibility $\pmod{\mathcal{W}}$, compressibility $\pmod{\mathcal{H}}$ and compressibility are all equivalent. A set $A \in \mathcal{B}$ is said to be decomposable $\pmod{n}$ if there exists $N \in \mathcal{n}$ such that $A - N$ is decomposable. If A is decomposable $\pmod{\mathcal{W}}$ then A is decomposable $\pmod{n}$ since $\mathcal{W} \subseteq \mathcal{n}$. If $\mathcal{B}$ is countably generated then every non-empty set in $\mathcal{B}$ is decomposable $\pmod{\mathcal{W}}$ hence also decomposable $\pmod{n}$.

The main result of this paper is the following:

2.14 Theorem. *If X is incompressible and every set in $\mathcal{B}$ is decomposable $\pmod{\mathcal{W}}$ then there exists a function m on $\mathcal{B} \times X$ such that*

- (i) $m(A, x) \geq 0$, $m(X, x) = 1$ and $m(A, x)$ is measurable in x
- (ii) $m(A, x) = m(T^{-1}A, x) = m(A, T^{-1}x) \pmod{\mathcal{H}}$
- (iii) $m(A, x) \equiv 0 \pmod{\mathcal{H}} \Leftrightarrow A \in \mathcal{H}$
- (iv) whenever $A_1, A_2, A_3, \dots$ are pairwise disjoint set in $\mathcal{B}$

$$m\left(\bigcup_{i=1}^{\infty} A_i, x\right) = \sum_{i=1}^{\infty} m(A_i, x) \pmod{\mathcal{H}}.$$

- (v) If $E \in \mathcal{B}$ is an invariant set then for all $A \in \mathcal{B}$ and $x \in E$

$$m(A \cap E, x) = m(A, x) \pmod{\mathcal{H}}.$$

A proof of this theorem will emerge as a consequence of a series of discussions which follow.

2.15 We say that B is a copy of $A \pmod{n}$ if $A \sim B \pmod{n}$. If inside A there are two or more pairwise disjoint copies of $A \pmod{n}$ then A is compressible $\pmod{n}$. If X accommodates infinitely many pairwise disjoint copies of $A \pmod{n}$ then sA is compressible $\pmod{n}$. We write $A \leq B \pmod{n}$ if there is a $C \subseteq B$ such that $A \sim C \pmod{n}$. If in addition $s(B - C) = sB \pmod{n}$, then we write $A < B \pmod{n}$. If $A < A \pmod{n}$ then $sA < A \pmod{n}$ and $sA \sim A \pmod{n}$. A proof of this can be found in ([8], 5-7).

2.16 If A and B are sets in $\mathcal{B}$ for which one of the relations $A \sim B$, $A \leq B$, $A < B$ holds then the relation also holds for $A \cap E$ and $B \cap E$ for any T -invariant set E in $\mathcal{B}$.

2.17 Notation. Suppose $A, B \in \mathcal{B}$ and B admits n or more pairwise disjoint copies of $A \pmod{n}$ then we can write $B = B_1 \cup B_2 \cup \dots \cup B_n \cup R$, where $B_1, B_2, \dots, B_n, R$ are pairwise disjoint and $A \sim B_1 \sim \dots \sim B_n$. We express this by writing $B = n \odot A \oplus R$, $R \subseteq B$. We also write $n \odot A \subseteq B$ to express the fact that B admits n or more pairwise disjoint copies of $A \pmod{n}$.

2.18 In the rest of this section we will deal only with the σ -ideal $\mathcal{H}$ and the relations $\sim, <, \leq$ will be with respect to this σ -ideal. We will therefore drop the qualification $\pmod{\mathcal{H}}$ after these symbols.

2.19 Lemma. Given $A, B \in \mathcal{B}$ we can write

$$A = E_1 \cup E_2 \cup E_3, \quad sE_i \cap sE_j = \emptyset \text{ if } i \neq j$$

$$B = F_1 \cup F_2 \cup F_3, \quad sF_i \cap sF_j = \emptyset \text{ if } i \neq j$$

such that modulo $\mathcal{H}$ we have

$$(i) \quad F_1 \prec E_1 \quad E_2 \sim F_2 \quad E_3 \prec F_2$$

$$(ii) \quad sE_1 \cap B = F_1, sE_2 \cap B = F_2, sF_2 \cap A = E_2, sF_3 \cap A = E_3,$$

Moreover such a decomposition is unique (mod $\mathcal{H}$).

Proof. We define sets $A(n), B(n), A_n, B_n, n = 0, 1, 2, 3, \dots$ by the following inductive procedure

$$A(0) = A$$

$$B(0) = B$$

$$A_0 = A(0) \cap B(0)$$

$$B_0 = B(0) \cap A(0)$$

$$A(1) = A(0) - A_0$$

$$B(1) = B(0) - B_0$$

$$A_1 = A(1) \cap TB(1)$$

$$B_1 = T^{-1}A_1 = T^{-1}A(1) \cap B(1)$$

$$\vdots$$

$$\vdots$$

$$A(n+1) = A(n) - A_n$$

$$B(n+1) = B(n) - B_n$$

$$A_{n+1} = A(n+1) \cap T^{n+1}B(n+1) \quad B_{n+1} = T^{-n-1}A(n+1) \cap B(n+1)$$

$$\vdots$$

$$\vdots$$

The sets $A(n)$ are decreasing and we write A_∞ for their intersection. Similarly the sets $B(n)$ are decreasing and we write B_∞ for their intersection. From our construction we have

$$A = \bigcup_{i=1}^{\infty} A_i \cup A_\infty \quad A_i \cap A_j = \emptyset \text{ if } i \neq j$$

$$B = \bigcup_{i=1}^{\infty} B_i \cup B_\infty \quad B_i \cap B_j = \emptyset \text{ if } i \neq j$$

$$T^i B_i = A_i \quad \text{for } i < \infty$$

so that $\bigcup_{i=1}^{\infty} A_i \sim \bigcup_{i=1}^{\infty} B_i$. Our construction also shows that for $k \geq 1$, $T^k B_\infty$ is disjoint from A_∞ so that $\bigcup_{k=1}^{\infty} T^k B_\infty$ is disjoint from A_∞ . In view of remarks in 2.12 $sA_\infty \cap sB_\infty \in \mathcal{W}$ hence also in $\mathcal{H}$ since $\mathcal{W} \subseteq \mathcal{H}$. We write

$$E_1 = A \cap sA_\infty \quad F_1 = B \cap sA_\infty$$

$$E_3 = A \cap sB_\infty \quad F_3 = B \cap sB_\infty$$

$$E_2 = A - (E_1 \cup E_3) \quad F_2 = B - (F_1 \cup F_3).$$

Now

$$F_1 = B \cap sA_\infty = \bigcup_{n=1}^{\infty} (B_i \cap sA_\infty) \cup (sA_\infty \cap B_\infty) = \bigcup_{n=1}^{\infty} (B_i \cap sA_\infty) \pmod{\mathcal{W}}$$

$$E_1 = A \cap sA_\infty = A_\infty \cup \bigcup_{n=1}^{\infty} (A_i \cap sA_\infty).$$

Hence we see that $F_1 < E_1$. Similarly $E_3 < F_3$.

Finally

$$F_2 = B - (F_1 \cup F_3) = \bigcup_{n=1}^{\infty} B_i - s(A_\infty \cup B_\infty)$$

$$E_2 = A - (E_1 \cup E_3) = \bigcup_{n=1}^{\infty} A_i - s(A_\infty \cup B_\infty)$$

so that $E_2 \sim F_2$. This proves (i). Now (ii) follows from our construction. It remains to prove the uniqueness part. To this end suppose that $A = E'_1 \cup E'_2 \cup E'_3$, $B = F'_1 \cup F'_2 \cup F'_3$ is another decomposition of A and B as in the lemma. Then the sets $E_1 \cap E'_2$ and $F_1 \cap F'_2$ satisfy the relations $E_1 \cap E'_2 \sim F_1 \cap F'_2$ and also $F_1 \cap F'_2 < E_1 \cap E'_2$ which is a contradiction unless $E_1 \cap E'_2$, $F_1 \cap F'_2$ belong to $\mathcal{H}$. Indeed

$$F_1 \cap F'_2 = sE_1 \cap B \cap F'_2 = sE_1 \cap F'_2 \sim sE_1 \cap E'_2 = sE_1 \cap A \cap E'_2 = E_1 \cap E'_2$$

$$F_1 \cap F'_2 = F_1 \cap B \cap sE'_2 = F_1 \cap sE'_2 < E_1 \cap sE'_2 = E_1 \cap A \cap sE'_2 = E_1 \cap E'_2.$$

Similarly we can show that if $i \neq j$, $E_i \cap E'_j$, $F_i \cap F'_j$ belong to $\mathcal{H}$. Thus the decomposition of A and B satisfying the requirements of the lemma is unique (mod $\mathcal{H}$). (q.e.d.)

2.20 COROLLARY

Given $A, B \in \mathcal{B}$ we can write

$$A = E \cup F \quad sE \cap sF = \emptyset$$

$$B = G \cup H \quad sG \cap sH = \emptyset$$

such that (mod $\mathcal{H}$) (i) $G < E \leq F \leq H$, (ii) $sE \cap B = G$, $sH \cap A = F$. Moreover such a decomposition is unique (mod $\mathcal{H}$).

Proof. We set $E = E_1$, $F = E_2 \cup E_3$, $G = F_1$, $H = F_2 \cup F_3$. Then G and H satisfy (i) and (ii). The uniqueness follows as before. (q.e.d.)

2.21 COROLLARY

If $A \in \mathcal{B} - \mathcal{H}$ is decomposable then a decomposition $A = C \cup D$, $C \cap D = \emptyset$, $sC = sD = sA$ (mod $\mathcal{H}$) can be so chosen that $C \leq D$.

2.22 A repeated application of corollary 2.20 gives us the following important lemma.

2.23 Lemma. Given $P, Q \in \mathcal{B}$, $P \leq Q$ we can decompose P and Q as follows

$$(i) \quad P = \bigcup_{i=1}^{\infty} P_i \cup P_\infty, sP_i \cap sP_j = \emptyset \quad \text{for } i \neq j$$

$$(ii) \quad Q = \bigcup_{i=1}^{\infty} Q_i \cup Q_\infty, sQ_i \cap sQ_j = \emptyset \quad \text{for } i \neq j$$

(iii) for each $i < \infty$, $sP_i = sQ_i$ and $Q_i = i \odot P_i \oplus R_i$, $R_i < P_i$

(iv) $P_\infty \in \mathcal{H}$.

Moreover such a decomposition of P and Q is unique (mod $\mathcal{H}$).

Proof. Since $P \leq Q$ we can write $Q = 1 \odot P \oplus R$, $R \subseteq Q$. We apply corollary 2.20 with $A = P$ and $B = R$ and let $P = E \cup F$, $R = G \cup H$ be the decompositions of P and R provided by the corollary. Put $P_1 = E = sE \cap P$, $Q_1 = sE \cap Q$, $P'_1 = F = P - P_1$, $Q'_1 = Q - Q_1$. We have

$$P = P_1 \cup P'_1 \quad sP_1 \cap sP'_1 = \emptyset$$

$$Q = Q_1 \cup Q'_1 \quad sQ_1 \cap sQ'_1 = \emptyset$$

$sP_1 = sQ_1$, $Q_1 = 1 \odot P_1 \oplus R_1$, $R_1 < P_1$, $2 \odot P'_1 \subseteq Q'_1$. Suppose we have obtained $P_1, \dots, P_n, P'_n, Q_1, \dots, Q_n, Q'_n$ such that

$$\left. \begin{aligned} P &= \bigcup_{i=1}^n P_i \cup P'_n, sP_i \cap sP_j = \emptyset, i \neq j, sP_i \cap sP'_n = \emptyset \text{ for all } i \\ Q &= \bigcup_{i=1}^n Q_i \cup Q'_n, sQ_i \cap sQ_j = \emptyset, i \neq j, sQ_i \cap sQ'_n = \emptyset \text{ for all } i \\ sQ_i &= sP_i, Q_i = i \odot P_i \oplus R_i, R_i < P_i, (n+1) \odot P'_n \subseteq Q'_n \end{aligned} \right\} *$$

Since $(n+1) \odot P'_n \subseteq Q'_n$ we can write

$$Q'_n = (n+1) \odot P_n \oplus R'_n, R'_n \subseteq Q'_n.$$

We now apply corollary 2.20 with $A = P'_n$, $B = R'_n$. If $P'_n = E_n \cup F_n$, $R'_n = G_n \cup H_n$ be the decomposition of P'_n , R'_n provided by the corollary, then we set

$$P_{n+1} = E_n = P'_n \cap sE_n \quad P'_{n+1} = P'_n - P_{n+1} = F_n$$

$$Q_{n+1} = Q'_n \cap sE_n \quad Q'_{n+1} = Q'_n - Q_{n+1}.$$

The sets $P_1, P_2, \dots, P_{n+1}, P'_{n+1}, Q_1, Q_2, \dots, Q_{n+1}, Q'_{n+1}$ satisfy (*) with n replaced by $(n+1)$. We observe also that for each n , $P'_{n+1} \subseteq P'_n$, $Q'_{n+1} \subseteq Q'_n$, $(n+1) \odot P'_n \subseteq Q'_n$. If we set $P_\infty = \bigcap_{n=1}^\infty P'_n$ and $Q_\infty = \bigcap_{n=1}^\infty Q'_n$ then Q_∞ accommodates infinitely many pairwise disjoint copies of P_∞ , whence $P_\infty \in \mathcal{H}$. We thus have

$$P = \bigcup_{n=1}^\infty P_i \cup P_\infty, sP_i \cap sP_j = \emptyset \quad \text{if } i \neq j$$

$$Q = \bigcup_{n=1}^\infty Q_i \cup Q_\infty, sQ_i \cap sQ_j = \emptyset \quad \text{if } i \neq j$$

$sP_i = sQ_i$, $Q_i = i \odot P_i \oplus R_i$, $R_i < P_i$, for $i < \infty$, $P_\infty \in \mathcal{H}$ which is the decomposition required by the lemma. The uniqueness (mod $\mathcal{H}$) of the decomposition follows as before.

Remark. If $E \in \mathcal{B}$ is invariant then the above lemma then the decomposition of $E \cap P, Q$ provided by the lemma will be

$$E \cap P = \bigcup_{i=1}^{\infty} (E \cap P_i) \cup (E \cap P_{\infty})$$

$$Q = \bigcup_{i=1}^{\infty} (E \cap Q_i) \cup \left(Q - \bigcup_{i=1}^{\infty} E \cap Q_i \right).$$

2.24 Let $A, B \in \mathcal{B}$ and let

$$A = E \cup P \quad sE \cap sP = \emptyset$$

$$B = G \cup Q \quad sG \cap sQ = \emptyset$$

$G < E, P \leq Q$ be the decomposition of A and B provided by corollary 2.20. Let $(P_i) 1 \leq i \leq \infty, Q_i 1 \leq i \leq \infty$ be the decompositions of the pair P, Q provided by lemma 2.23. If $A \in \mathcal{B} - \mathcal{H}$ then we define

$$\left[\frac{B}{A} \right](x) = \begin{cases} i & \text{if } x \in sP_i \quad i < \infty \\ 0 & \text{otherwise.} \end{cases}$$

In view of the uniqueness (mod $\mathcal{H}$) of the decompositions provided by corollary 2.20 and lemma 2.23 the function $[B/A]$ is unambiguously defined (mod $\mathcal{H}$). We further observe that

$$(i) \quad C \sim B \Rightarrow \left[\frac{C}{A} \right] = \left[\frac{B}{A} \right] \pmod{\mathcal{H}}$$

$$(ii) \quad C \sim A \Rightarrow \left[\frac{B}{C} \right] = \left[\frac{B}{A} \right] \pmod{\mathcal{H}}$$

$$(iii) \quad \text{if } A \leq B \text{ then } \left[\frac{A}{C} \right] \leq \left[\frac{B}{C} \right] \pmod{\mathcal{H}} \text{ whereas if}$$

$$C \leq D \text{ then } \left[\frac{A}{C} \right] \geq \left[\frac{A}{D} \right] \pmod{\mathcal{H}}$$

$$(iv) \quad \text{if } E \text{ is } T\text{-invariant then for } x \in E \quad \left[\frac{B}{A} \right](x) = \left[\frac{B \cap E}{A} \right](x) \pmod{\mathcal{H}}$$

2.25 PROPOSITION

If E be the set of points x where $[B/A](x) < [C/A](x)$, then E is T -invariant and $E \cap B \leq E \cap C$.

Proof. Clearly the set E is T -invariant. Let $i = [B/A](x)$ and $j = [C/A](x)$ and assume that $i < j$. Let I be the set of points x where $[C/A](x) = j$ and $[B/A](x) = i$. Then I is T -invariant and $B \cap I = i \odot (A \cap I) \oplus R, R < A \cap I$ whereas $C \cap I = j \odot (A \cap I) \oplus S, S < A \cap I$. Since $i < j$ and $R < A \cap I$, we see that $B \cap I < C \cap I$. (q.e.d.)

2.26 Lemma. (i) $[A/B][B/C] \leq [A/C] < ([A/B] + 1)([B/C] + 1) \pmod{\mathcal{H}}$

(ii) $[A/C] + [B/C] \leq [A \cup B/C] < [A/C] + [B/C] + 2 \pmod{\mathcal{H}}$ where A and B are disjoint.

Proof. Let $[A/B](x) = i$, $[B/C](x) = j$, $[A/C](x) = k$ and let E be the set of points where the set E is compressible since we wish to prove the result (mod $\mathcal{H}$). Assume therefore that the set E is incompressible. Now $E \cap B$ admits j pairwise disjoint copies of $E \cap C$ and $E \cap A$ admits i pairwise disjoint copies of $E \cap B$ whence $E \cap A$ admits at least ij pairwise disjoint copies of $E \cap C$. Thus $k =$ number of copies of $E \cap C$ inside $E \cap A$ is bigger than or equal to ij . This proves first half of (i). To prove the second half assume that $k \geq (i+1)(j+1)$. Then $E \cap A$ admits at least $(i+1)(j+1)$ pairwise disjoint copies $E \cap C$. Each collection of $(j+1)$ such copies of $(C \cap E)$ will admit a copy of $B \cap E$, since $[B/C] = j$ on E . Thus $E \cap A$ admits at least $(i+1)$ pairwise disjoint copies of $E \cap B$. Since $[A/B] = i$ on E and E is incompressible we have a contradiction. Hence $k < (i+1)(j+1)$ and (i) is proved.

Proof of (ii). If $x \notin A \cap B$ then at least one of the integers $[A/C](x)$, $[B/C](x)$ is zero and the other is equal to $[(A \cup B)/C](x)$ so that the inequality is valid for $x \notin A \cap B$. Assume therefore that $x \in A \cap B$ and let $i = [A/C](x)$ and $j = [B/C](x)$. Let E be the set of points where $[A/C] = i$ and $[B/C] = j$. We can ignore the pairs i, j where E is compressible. Now

$$E \cap A = i \odot (E \cap C) \oplus R, \quad R < E \cap C$$

$$E \cap B = j \odot (E \cap C) \oplus S, \quad S < E \cap C.$$

Since $A \cap B = \emptyset$. We have

$$(E \cap A) \cup (E \cap B) = (i+j) \odot (E \cap C) \oplus (R \cup S), \quad R < E \cap C, S < E \cap C.$$

Thus $[(A \cup B)/C]$ is at least equal to $(i+j)$ on E but it cannot exceed $(i+j+1)$ since $R \cup S$ can admit at most one copy of $E \cap C$ inside it. Thus we have $(i+j) \leq [(A \cup B)/C] < i+j+2$ on E , which proves (ii). (q.e.d.)

2.27 Let X be incompressible. A sequence $(F_n)_{n=1}^{\infty}$ of sets in $\mathcal{B}$ is said to be fundamental if for all n , $sF_n = X$ and $[F_n/F_{n+1}](x) \geq 2 \pmod{\mathcal{H}}$. We note that the requirement $[F_n/F_{n+1}](x) \geq 2 \pmod{\mathcal{H}}$ implies that $F_{n+1} < F_n \pmod{\mathcal{H}}$. If X is incompressible and every set in $\mathcal{B} - \mathcal{H}$ is decomposable then there exists a fundamental sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{B}$. This follows on setting $F_1 = X$ and using successively the decomposability in the manner described in corollary 2.21.

2.28 *Lemma.* Let $(F_n)_{n=1}^{\infty}$ be a fundamental sequence in $\mathcal{B}$. Then for $A \in \mathcal{B}$, $\lim_{n \rightarrow \infty} [A/F_n](x)$ exists (mod $\mathcal{H}$). The limit is equal to zero on $X - sA$ and equal to ∞ on sA .

Proof. Since $F_{n+1} < F_n$, $[A/F_n] \leq [A/F_{n+1}]$. Further by lemma 2.26 (i)

$$\left[\frac{A}{F_{n+k}} \right] \geq \left[\frac{A}{F_n} \right] \left[\frac{F_n}{F_{n+k}} \right] \geq 2^k \left[\frac{A}{F_n} \right] \pmod{\mathcal{H}}.$$

Hence on the set of points x where

$$\left[\frac{A}{F_n} \right](x) \neq 0, \quad \left[\frac{A}{F_{n+k}} \right](x) \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Thus either $[A/F_n](x) = 0$ for all n or $[A/F_n](x) \rightarrow \infty$ as $n \rightarrow \infty \pmod{\mathcal{H}}$, and so the limit in question exists. If $A \in \mathcal{H}$, then the limit is zero $\pmod{\mathcal{H}}$. Now if $A \in \mathcal{B} - \mathcal{H}$, then $[F_{n+1}/A] \leq [F_n/A]$ since $F_{n+1} \subseteq F_n$. Further

$$\left[\frac{F_1}{A} \right] \geq \left[\frac{F_1}{F_n} \right] \left[\frac{F_n}{A} \right] \geq 2^n \left[\frac{F_n}{A} \right].$$

Since $[F_1/A]$ is finite valued and $[F_n/A]$ is non-negative integer valued, we see that $[F_n/A](x) \rightarrow 0$ as $n \rightarrow \infty$.

Let E_n be the set of points x where $[F_n/A](x)$ is zero. Then $E_{n+1} \supseteq E_n$ and their union is $X \pmod{\mathcal{H}}$. Now $A \cap E_n \rightarrow A \pmod{\mathcal{H}}$ and since $[F_n/A]$ vanishes on E_n , $[A/F_n] > 0 \pmod{\mathcal{H}}$ on $s(A \cap E_n)$ whenever $A \cap E_n \in \mathcal{B} - \mathcal{H}$. From the first part of this lemma we see that $[A/F_n](x) \rightarrow \infty$ on $s(A \cap E_k)$. Since this holds for all k , $\lim [A/F_n](x) = \infty$ for $x \in sA \pmod{\mathcal{H}}$. (q.e.d.)

For any two sets $A, B \in \mathcal{B}$ and any $C \in \mathcal{B} - \mathcal{H}$, the ratio $[A/C](x) \div [B/C](x)$ will be assigned the value zero whenever the numerator is zero (even in the case where the denominator is also zero). If the numerator is non-zero and the denominator is zero, the ratio will be assigned the value $+\infty$.

2.29 Lemma. Let $(F_n)_{n=1}^\infty$ be a fundamental sequence in $\mathcal{B}$. Then for all $A, B \in \mathcal{B}$, $\lim_{n \rightarrow \infty} \{[A/F_n](x)\} / \{[B/F_n](x)\}$ exists $\pmod{\mathcal{H}}$ and it assumes non-zero finite value on $sA \cap sB \pmod{\mathcal{H}}$.

Proof. The numerator is zero on $X - sA$ and the denominator is zero on $X - sB$. Hence on $X - sA$ the limit is zero and on $X - sB$ the limit is zero or ∞ depending on the value of the numerator. If $sA \cap sB \in \mathcal{H}$ then there is nothing left to prove since the limit is claimed to exist $\pmod{\mathcal{H}}$. Assume therefore that $sA \cap sB \in \mathcal{B} - \mathcal{H}$. There is no loss of generality if we assume that $sA = sB$. Now by lemma 2.26 (i)

$$\begin{aligned} \left[\frac{A}{F_{i+j}} \right] &\leq \left(\left[\frac{A}{F_i} \right] + 1 \right) \left(\left[\frac{F_i}{F_{i+j}} \right] + 1 \right) \\ \left[\frac{B}{F_{i+j}} \right] &\geq \left[\frac{B}{F_i} \right] \left[\frac{F_i}{F_{i+j}} \right], \quad \text{whence} \\ \frac{\left[\frac{A}{F_{i+j}} \right]}{\left[\frac{B}{F_{i+j}} \right]} &\leq \frac{\left[\frac{A}{F_i} \right] + 1}{\left[\frac{B}{F_i} \right]} \cdot \frac{\left[\frac{F_i}{F_{i+j}} \right] + 1}{\left[\frac{F_i}{F_{i+j}} \right]} \leq \frac{\left[\frac{A}{F_i} \right] + 1}{\left[\frac{B}{F_i} \right]} \cdot \left(1 + \frac{1}{2^j} \right) \end{aligned}$$

we have

$$\limsup_{j \rightarrow \infty} \frac{\left[\frac{A}{F_{i+j}} \right](x)}{\left[\frac{B}{F_{i+j}} \right](x)} \leq \frac{\left[\frac{A}{F_i} \right](x) + 1}{\left[\frac{B}{F_i} \right](x)}, \quad \text{that is,}$$

$$\limsup_{n \rightarrow \infty} \frac{\left[\frac{A}{F_n} \right]}{\left[\frac{B}{F_n} \right]} \leq \frac{\left[\frac{A}{F_i} \right] + 1}{\left[\frac{B}{F_i} \right]}.$$

$$\limsup_{n \rightarrow \infty} \frac{\left[\frac{A}{F_n} \right](x)}{\left[\frac{B}{F_n} \right](x)} \leq \liminf_{n \rightarrow \infty} \frac{\left[\frac{A}{F_n} \right](x)}{\left[\frac{B}{F_n} \right](x)}, x \in sB - N.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{\left[\frac{A}{F_n} \right](x)}{\left[\frac{B}{F_n} \right](x)} \text{ exists and is finite valued (mod } \mathcal{H})$$

on $sA = sB = sA \cap sB$. Interchanging the role of A and B proves that the limit is non-zero on $sA \cap sB$. (q.e.d.)

$$2.30 \text{ For } A \in \mathcal{B} \text{ write } m(A, x) = \lim_{n \rightarrow \infty} \frac{\left[\frac{A}{F_n} \right](x)}{\left[\frac{X}{F_n} \right](x)}$$

where $(F_n)_{n=1}^\infty$ is a fundamental sequence in $\mathcal{B}$. The function $m(A, x)$ is measurable in x and in addition has the following properties:

- Lemma.* (i) $A \sim B \Rightarrow m(A, x) = m(B, x) \pmod{\mathcal{H}}$
(ii) $m(A, x) = m(A, T^{-1}x) \pmod{\mathcal{H}}$
(iii) $m(A, x) = \lim_{n \rightarrow \infty} [A/F_n](x)/[sA/F_n](x) \pmod{\mathcal{H}}$
(iv) $m(A, x) \equiv 0 \pmod{\mathcal{H}} \Leftrightarrow A \in \mathcal{H}, m(A, x) > 0 \pmod{\mathcal{H}}$ on sA
(v) If $A \cap B = \emptyset$ then $m(A \cup B, x) = m(A, x) + m(B, x), \pmod{\mathcal{H}}$
(vi) If E be the set of points where $m(A, x) < m(B, x)$, then $A \cap E \leq B \cap E$.

Proof. Properties (i), (ii) and (iii) follow immediately from the definition of $m(A, x)$. Property (iv) is proved in lemma 2.29. To prove (v) we note that (from lemma 2.26 (ii))

$$\frac{\left[\frac{A}{F_n} \right] + \left[\frac{B}{F_n} \right]}{\left[\frac{X}{F_n} \right]} \leq \frac{\left[\frac{A \cup B}{F_n} \right]}{\left[\frac{X}{F_n} \right]} \leq \frac{\left[\frac{A}{F_n} \right] + \left[\frac{B}{F_n} \right] + 2}{\left[\frac{X}{F_n} \right]}.$$

Since $[X/F_n] \geq 2^n$ we see on letting $n \rightarrow \infty$ that $m(A, x) + m(B, x) \leq m(A \cup B, x) \leq m(A, x) + m(B, x)$ which proves (v). To prove (vi) let E_n be the set of points x where

$$\frac{\left[\frac{A}{F_n} \right](x)}{\left[\frac{X}{F_n} \right](x)} < \frac{\left[\frac{B}{F_n} \right](x)}{\left[\frac{X}{F_n} \right](x)},$$

i.e., E_n is the set of points x where $[A/F_n](x) < [B/F_n](x)$. From the proposition 2.25, $E_n \cap A \leq E_n \cap B$. Now $E = \bigcup_{n=1}^{\infty} E \cap E_n = \bigcup_{n=1}^{\infty} G_n$, where $G_1, G_2, G_3, \dots$ is the usual disjointification of $E \cap E_1, E \cap E_2, \dots$, i.e., $G_n = E \cap E_n - \bigcup_{i=1}^{n-1} (E \cap E_i)$, $n = 1, 2, 3, \dots$. Each G_n is T -invariant and $G_n \subseteq E_n$, whence $G_n \cap A \leq G_n \cap B$. Since G_n 's are pairwise disjoint we see that $A \cap E \leq B \cap E$. This proves (vi). (q.e.d.)

2.31 Lemma. If $m(A, x) > \sum_{k=1}^{\infty} m(A_k, x) \pmod{\mathcal{H}}$ where $A_1, A_2, A_3, \dots$ are pairwise disjoint sets in $\mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \leq A$.

Proof. Since $m(A, x) < m(A_1, x)$, by 2.30 (vi) there is a set $B_1 \subseteq A$ such that $B_1 \sim A_1$. By 2.30 (v) $m(A, x) = m(A - B_1, x) + m(B_1, x) > \sum_{k=1}^{\infty} m(A_k, x)$. Since $m(A_1, x) = m(B_1, x)$ by 2.30 (i), we have $m(A - B_1, x) > \sum_{k=2}^{\infty} m(A_k, x)$. We apply the same argument to obtain a copy B_2 of A_2 in $A - B_1$. Proceeding thus we see that $\bigcup_{i=1}^{\infty} A_i \leq A$. (q.e.d.)

2.32 Lemma. If $A_1, A_2, A_3, \dots$ are pairwise disjoint sets in $\mathcal{B}$ then $m(\bigcup_{k=1}^{\infty} A_k, x) = \sum_{k=1}^{\infty} m(A_k, x) \pmod{\mathcal{H}}$.

Proof. If the function m is not countably additive in first variable then there exist $A_1, A_2, A_3, \dots$ in $\mathcal{B}$ pairwise disjoint such that $m(\bigcup_{i=1}^{\infty} A_i, x) \neq \sum_{i=1}^{\infty} m(A_i, x) \pmod{\mathcal{H}}$. In as much as $m(\bigcup_{i=1}^{\infty} A_i, x) \geq m(\bigcup_{i=1}^n A_i, x) = \sum_{i=1}^n m(A_i, x)$ we see that $m(\bigcup_{i=1}^{\infty} A_i, x) \neq \sum_{k=1}^{\infty} m(A_i, x)$ means that $m(\bigcup_{i=1}^n A_i, x) > \sum_{i=1}^n m(A_i, x)$ on an incompressible set in $\mathcal{B}$. There is no harm in assuming that $\bigcup_{i=1}^{\infty} A_i = X$, for otherwise we can simply replace A_1 by $A_1 \cup (X - \bigcup_{i=1}^{\infty} A_i)$. Now, since the set on which $m(\bigcup_{i=1}^{\infty} A_i, x) > \sum_{i=1}^{\infty} m(A_i, x)$ is incompressible, for some k , the set E of points x where $m(\bigcup_{i=1}^{\infty} A_i, x) - 1/k > \sum_{i=1}^{\infty} m(A_i, x)$ is incompressible. We can assume without loss of generality that $E = X$, otherwise we can treat E as our space X . Choose n such that $m(F_n) \leq 1/2^n < 1/2^k$. Then $m(X - F_n, x) \geq 1 - (1/2^n) > 1 - (1/k) = m(X, x) - (1/k)$. We have assumed that $\bigcup_{i=1}^{\infty} A_i = X$. Thus $m(X - F_n, x) > m(\bigcup_{i=1}^{\infty} A_i, x) - (1/k) > \sum_{i=1}^{\infty} m(A_i, x)$. By 2.31, $\bigcup_{i=1}^{\infty} A_i \leq X - F_n$. Since $sF_n = s(X - F_n) = X$ we see that X is compressible which is a contradiction. (q.e.d.)

2.33 If $E \in \mathcal{B}$ is T -invariant then for all $B \in \mathcal{B}$ and for $x \in E$

$$m(B, x) = m(B \cap E, x) \pmod{\mathcal{H}}.$$

This holds because from the definition of the quantity $[B/F_n](x)$ we note that for $x \in E$,

$$\left[\frac{B}{F_n} \right](x) = \left[\frac{B \cap E}{F_n} \right](x) \pmod{\mathcal{H}}.$$

Summing up we have

2.34 Theorem. Let X be incompressible $\pmod{\mathcal{W}}$ and assume that there exists a fundamental sequence in $\mathcal{B}$. Then there exists a function $m: \mathcal{B} \times X \rightarrow [0, 1]$ such that

- (i) $m(A, x)$ is measurable in x for all $A \in \mathcal{B}$, $m(X, x) = 1$
- (ii) $m(A, x) = m(T^{-1}A, x) = m(A, T^{-1}x) \pmod{\mathcal{H}}$
- (iii) $m(A, x) \equiv 0 \pmod{\mathcal{H}}$ if and only if $A \in \mathcal{H}$
- (iv) $m(\bigcup_{i=1}^{\infty} A_i, x) = \sum_{i=1}^{\infty} m(A_i, x) \pmod{\mathcal{H}}$ whenever $A_1, A_2, A_3, \dots$ are pairwise disjoint sets in $\mathcal{B}$.

(v) If $E \in \mathcal{B}$ is T -invariant then for all $B \in \mathcal{B}$ and $x \in E$, $m(B, x) = m(B \cap E, x) \pmod{\mathcal{H}}$.

The main theorem 2.14 follows because whenever every set in $\mathcal{B}$ is decomposable $\pmod{\mathcal{H}}$, then there exists a fundamental sequence in $\mathcal{B}$.

Remark 1. One can develop integration with respect to m in a natural fashion and prove the maximal ergodic theorem. If f is integrable with respect to m then the set

$$E = \left\{ x : \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \text{ does not converge as } n \rightarrow \infty \right\}$$

can be shown to be such that $m(E, x) \equiv 0 \pmod{\mathcal{H}}$, i.e., $E \in \mathcal{H}$. This yields 'universal' form of Birkhoff's ergodic theorem.

Remark 2. Given a system $(X, \mathcal{B}, \tau, T)$ one can form the σ -ideal $\mathcal{F}$ of those sets $B \in \mathcal{B}$ whose saturations are compressible with respect to τ . It seems that the above analysis can be carried out with respect to $\mathcal{F}$ and we can get the 'transition probability' m (with respect to $\mathcal{F}$) whenever X is incompressible $\pmod{\tau}$.

3. The case of standard Borel space

3.1 In this section we show that in case $(X, \mathcal{B})$ is a standard Borel space incompressible under T then there exists a probability measure on $\mathcal{B}$ invariant under T . We do this by putting together theorem 2.14 and some well known facts about measures and topologies on a standard Borel space. (see K R Parthasarathy's book on Probability Measures in Metric Spaces, Academic Press, 1967).

3.2 A result of Ramsay and Mackey states that we can assign to X a complete separable metric topology $\mathcal{T}$ such that T is a homeomorphism under this topology and the Borel σ -algebra generated by the topology coincides with $\mathcal{B}$ (see Weiss [11]). If we are given a countable collection $\tau \subseteq \mathcal{B}$, the topology can be so chosen that $\tau \subseteq \mathcal{T}$. (We will not need this latter fact.)

3.3 *Lemma.* Let X be a complete metric space with metric d . Let $(F_n)_{n=1}^{\infty}$ be a decreasing sequence of closed sets such that for each n , F_n is covered by finitely many closed balls each of diameter $< 1/n$. Then the set $F = \bigcap_{n=1}^{\infty} F_n$ is compact. It is non-empty whenever each F_n is non-empty.

Proof. It is clear from hypothesis that F is closed and totally bounded hence compact. If each F_n is non-empty we choose a point $x_n \in F_n$, $n = 1, 2, 3, \dots$. The sets $G_n = \text{closure of } \{x_n, x_{n+1}, \dots\}$ are non-empty closed and totally bounded with $G_{n+1} \subseteq G_n \subseteq F_n$. Hence $G = \bigcap_{n=1}^{\infty} G_n$ is non-empty and contained in F . (q.e.d.)

3.4 An outer measure μ^* on the power set 2^X of a metric space (X, d) is called a metric outer measure if $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ whenever $d(E, F) > 0$. If μ^* is a metric outer measure then every open set (and therefore every Borel set) is μ^* measurable. Thus μ^* is a measure on the class of Borel sets of X whenever μ^* is a metric outer measure on 2^X . (see Halmos 'Measure Theory', p. 48 exercises 8a, 8b).

We use Riesz-Mackey result quoted above to assign to $\mathcal{U}$ a countable separable metric topology $\mathcal{T}$ (with metric d) whose Borel sets form the class $\mathcal{B}$ and under which T is a homeomorphism. Let $\mathcal{U}$ be a countable open base for $\mathcal{T}$. The collection obtained by taking finite unions and finite intersections of sets in the collection $\bigcup_{n=-\infty}^{\infty} \{T^n U : U \in \mathcal{U}\}$ is again a countable open base for $\mathcal{T}$ which is closed under finite unions, finite intersections, and application of T . We therefore assume that $\mathcal{U}$ itself has these properties. Let $\bar{\mathcal{U}}$ denote the closures of sets in $\mathcal{U}$ and $\mathcal{A}$ the algebra generated by $\mathcal{U} \cup \bar{\mathcal{U}}$ which is also countable. Let $\{\bar{U}_{1n}, \bar{U}_{2n}, \bar{U}_{3n}, \dots\}$ denote the collection of members in $\bar{\mathcal{U}}$ of diameter $< 1/n, n = 1, 2, 3, \dots$. We know that

- (i) $m(A \cup B, x) = m(A, x) + m(B, x) \pmod{\mathcal{H}}$ whenever $A \cap B = \emptyset, A, B \in \mathcal{A}$
- (ii) for each $n, \lim_{k \rightarrow \infty} m(\bigcup_{i=1}^k \bar{U}_{in}, x) = m(X, x) = 1 \pmod{\mathcal{H}}$

Now (i) and (ii) above are countable number of conditions, hence there exists a T -invariant set $N \in \mathcal{H}$ such that for all $x \in X - N$ we have

- (a) $m(A \cup B, x) = m(A, x) + m(B, x)$ whenever $A \cap B = \emptyset, A, B \in \mathcal{A}$
- (b) for each $n, m(\bigcup_{i=1}^k \bar{U}_{in}, x) \rightarrow 1$ as $k \rightarrow \infty$.

Fix an $x \in X - N$ and write $m(A, x) = m(A)$ for $A \in \mathcal{A}$. Define the outer measure m^* on the power set of X by setting for any $B \subseteq X$

$$m^*(B) = \inf \left\{ \sum_{i=1}^{\infty} m(U_i) : B \subseteq \bigcup_{i=1}^{\infty} U_i, U_i \in \mathcal{U} \text{ for all } i \right\}$$

We note that m^* is T -invariant and bounded by 1.

3.6 Lemma. $m^*(X) = 1$.

Proof. We show that given $\varepsilon > 0$ there exists a compact set $F = F_\varepsilon$ such that $m^*(F) \geq 1 - \varepsilon$. From (b) above we can get for each n a k_n such that $m(\bigcup_{i=1}^{k_n} \bar{U}_{in}) > 1 - (\varepsilon/2^{n+1})$. If $F_n = \bigcap_{j=1}^n \bigcup_{i=1}^{k_j} \bar{U}_{ij}$, then for all n , (i) $F_n \supseteq F_{n+1}$, (ii) F_n is closed, (iii) F_n is covered by finitely many closed balls of diameter $< 1/n$, (iv) $F_n \in \mathcal{A}$ (v) $m(F_n) > 1 - \varepsilon$. The set $F = \bigcap_{n=1}^{\infty} F_n$ is compact by lemma 3.3. We show that $m^*(F) > 1 - \varepsilon$. Fix $\eta > 0$ and let $U_1, U_2, U_3, \dots$ be sets in $\mathcal{U}$ covering F such that $\sum_{i=1}^{\infty} m(U_i) - \eta < m^*(F)$. Since F is compact there is an integer p such that $F \subseteq \bigcup_{i=1}^p U_i \in \mathcal{U}$ (since $\mathcal{U}$ is closed under finite unions). By lemma 3.3 we conclude that for some integer $q, F_q \subseteq \bigcup_{i=1}^p U_i$. Finally $1 - \varepsilon - \eta < m(F_q) - \eta < m(\bigcup_{i=1}^p U_i) - \eta < \sum_{i=1}^{\infty} m(U_i) - \eta < m^*(F)$. Since $\eta > 0$ is arbitrary we see that $m^*(F) \geq 1 - \varepsilon$. (q.e.d.)

3.7 Lemma. If $K \subseteq X$ is compact then the restriction of m^* to K is a metric outer measure.

Proof. Let $E, F \subseteq K$ be positive distance apart. Since $\bar{E}, \bar{F}$ are compact and positive distance apart we can find $U, V \in \mathcal{U}, U \cap V = \emptyset$ such that $E \subseteq U, F \subseteq V$. Fix $\varepsilon > 0$ and let $W_1, W_2, \dots$ be sets in $\mathcal{U}$ covering $E \cup F$ such that $\sum_{n=1}^{\infty} m(W_n) - \varepsilon < m^*(E \cup F) \leq m^*(E) + m^*(F)$. Now $W_n \cap U, n = 1, 2, 3, \dots$ cover E and the sets $W_n \cap V, n = 1, 2, 3, \dots$ cover F . Moreover these sets belong to $\mathcal{U}$.

$$\begin{aligned}
m^*(E) + m^*(F) - \varepsilon &\leq \sum_{n=1}^{\infty} m(W_n \cap U) + \sum_{n=1}^{\infty} m(W_n \cap V) - \varepsilon \\
&\leq \sum_{n=1}^{\infty} m(W_n) - \varepsilon \\
&\leq m^*(E \cup F) \leq m^*(E) + m^*(F).
\end{aligned}$$

Thus if E and F are positive distance apart in K then $m^*(E \cup F) = m^*(E) + m^*(F)$, i.e., m^* is a metric outer measure on K . (q.e.d.)

By a result quoted earlier every Borel subset of K is m^* measurable and m^* is a countably additive measure on the Borel field of K . Moreover if $A \subseteq K$ is m^* measurable and if $TA \subseteq K$, then $m^*(A) = m^*(TA)$. Thus the collection of m^* measurable sets contain all compact sets and by lemma 3.3 there exists a σ -compact set $Y \subseteq X$ such that $m^*(Y) = 1$. This set Y can be made to be T -invariant by replacing Y by $\bigcup_{n=-\infty}^{\infty} T^n Y$. We thus see that m^* is a T -invariant probability measure on Borel subsets of X , i.e. on $\mathcal{B}$.

Remark 1. If n_μ denotes the class of μ null sets of a T -invariant probability measure μ on $(X, \mathcal{B})$, then $n_\mu \supseteq \mathcal{H}$. We see now that $\mathcal{H} = \bigcap n_\mu$ where intersection is taken over all T -invariant probability measures μ .

Remark 2. If $X = \bigcup X_i, i \in I$ an indexing set, be a partition of X generated by countable number of T -invariant Borel sets in X , then incompressibility of X implies the incompressibility of at least one of the X_i . This follows on taking the regular conditional probability of a T -invariant probability measure on X with respect to the partition.

Remark 3. Let $\mathcal{A} = \{A_1, A_2, A_3, \dots\}$ be a countable algebra generating $\mathcal{B}$ and consider the measurable map $X \rightarrow [0,1]^{\mathcal{A}}$ given by

$$x \rightarrow (m(A_1, x), m(A_2, x), \dots).$$

This map gives a countably generated measurable partition of X into T -invariant sets. A typical element of this partition is of the form $C_x = \{y: m(A_i, y) = m(A_i, x) \text{ for all } A_i \in \mathcal{A}\}$. The incompressible members of this partition are uniquely ergodic under T and every ergodic T -invariant probability measure is supported on some member of this partition.

4. Historical background and Hopf's theorem

The question of existence of invariant integrals (equivalently finite invariant measures) was discussed by G D Birkhoff and P A Smith in their expository paper 'Structure analysis of surface transformations' *Journ. de Math. Tome VII – Fasc IV, 1928*, pp. 345–379. Many elementary and basic ideas of classical ergodic theory are set forth in this paper, although in the setting of continuous or analytic invertible maps of a surface. Let us briefly review the contents of §4 of this paper of Birkhoff and Smith.

arbitrarily small area. This result is an intuitive statement of their result which we explain below.

They begin by introducing a function $\varphi(e)$, e being a measurable set of S , defined as follows: Let e be divided into a finite number of mutually exclusive measurable sets

$$e = \bigcup_i \delta_i, \quad \delta_i \cap \delta_j = \emptyset \text{ for } i \neq j.$$

Then $\varphi(e)$ is the lower bound of the sum

$$\sum_i m(T^{n_i} \delta_i) \quad (m = \text{surface area})$$

with respect to all possible methods of subdivision of e into finite number of measurable sets, and all possible choices of integers n_i . Here use is made of the fact that the property of measurability is preserved under analytic transformation.

It is clear that function φ may be identically zero, in which case, S is compressible into an arbitrarily small area. This happens, for example, when T is an analytic transformation of a sphere such that each circle parallel to the equator closes down on the north (or south) pole on indefinite iteration of T (or T^{-1}). On the other hand, for a transformation which preserves the area measure m , we have $\varphi(e) = m(e)$.

In any case, it follows immediately from the definition, that $\varphi(e) \leq m(e)$, and hence $\varphi(e)$ is bounded and absolutely continuous with respect to m . The importance of φ is due to the following theorem of Birkhoff and Smith.

Theorem. φ is countably additive and invariant under T . Moreover there exists a finite, non-trivial T -invariant measure on S absolutely continuous with respect to m if and only if $\varphi(S) > 0$.

It is interesting to note that Birkhoff and Smith are careful to qualify their invariant integrals as invariant integrals of a certain type by which they seem to mean invariant integrals which are finite non-trivial and absolutely continuous with respect to m . The question of finding necessary and sufficient conditions for the existence of finite non-trivial invariant integrals, not necessarily absolutely continuous with respect to m , seems to have remained uninvestigated.

On the suggestion of Birkhoff, the question of the existence of a finite invariant measure was further taken up by Hopf in his paper [5] where the notion of compressibility due to Birkhoff and Smith was modified as follows: Consider a measure space $(X, \mathcal{B}, \mu)$ where we assume that $(X, \mathcal{B})$ is standard Borel and $\mu(X) = 1$. Let $T: X \rightarrow X$ be one-one onto measurable map which preserves μ -null sets. Two sets $A, B \in \mathcal{B}$ are said to be equivalent by countable decomposition (mod μ) if we can find μ -null sets $M, N \in \mathcal{B}$ such that $A - M, B - N$ are equivalent by countable decomposition in our sense, i.e., in the sense of definition 2.3. A set $A \in \mathcal{B}$ of positive measure is said to be compressible in the sense of Hopf if we can decompose A into pairwise disjoint sets $C, D \in \mathcal{B}$, each of positive measure and further A is equivalent to C by countable decomposition (mod μ). It is clear that if a set A is compressible in our sense then it is compressible in the sense of Hopf. But the converse need not hold. For example

let $X = \mathbb{R} \cup S^1$ with the usual Borel structure and a probability measure having same null sets as Lebesgue measure on $\mathbb{R}$ and S^1 . (Here $S^1 =$ circle group). Let T act on X by translation by one on $\mathbb{R}$ and irrational rotation on S^1 . Then X is compressible in the sense of Hopf with respect to the measure μ but not in our sense.

Theorem. (Hopf). *If X is incompressible in Hopf's sense then there exists a unique T -invariant probability measure on $\mathcal{B}$ having same null sets as μ .*

Hopf's proof of the theorem was considered difficult. Simplified proofs and alternative necessary and sufficient conditions for the existence of finite invariant measure equivalent to a given one for a non-singular transformation were therefore sought. The best known is the theorem of Hajian and Kakutani which says that a non-singular T on the measure space $(X, \mathcal{B}, \mu)$ admits a finite equivalent T -invariant measure if and only if there is no weakly wandering set of positive measure, where a set $A \in \mathcal{B}$ is said to be weakly wandering if there exists a sequence $(n_k)_{k=1}^{\infty}$ of integers such that iterates $T^{n_k}A$ are pairwise disjoint (see [3]).

Hopf's theorem follows easily from the above quoted result of Hajian and Kakutani because the existence of a weakly wandering set of positive measure immediately implies compressibility (mod μ) in the sense of Hopf. We can use theorem 2.14 of this paper to prove Hopf's theorem as follows. If X is incompressible in the sense of Hopf (with respect to μ) then X is incompressible in the sense of definition 2.4. Also since $(X, \mathcal{B})$ is standard Borel every non-empty set in $\mathcal{B}$ is decomposable (mod $\mathcal{W}$). Thus the hypothesis of theorem 2.14 is satisfied. If m be as in theorem 2.14 we write

$$P(A) = \int_X m(A, x) d\mu.$$

It is easy to show, using properties of m , that P is a T -invariant probability measure having same null sets as μ and that it is unique.

Since we have mentioned the theorem of Hajian and Kakutani, (*Trans. Amer. Math. Soc.* **110** (1964) 136–151) it is natural to ask whether compressibility of X under T implies the existence of a weakly wandering set $W \in \mathcal{B}$ such that $sW = X$. One may assume, if necessary, that $(X, \mathcal{B})$ is a standard Borel space. A related question is whether a compressing (in the sense of definition 2.4) homeomorphism of a Polish space admits a weakly wandering non-empty open set. It is not necessarily required that the saturation of such an open set be all of X .

Another question which can be formulated is as follows: Suppose f is a non-negative measurable function on $(X, \mathcal{B})$ and T a Borel automorphism on X . Can one define a notion “ X is incompressible with respect to f ” which implies that whenever X is thus incompressible there exists a σ -finite measure m on $\mathcal{B}$ quasi-invariant under T such that $dm_T/dm = f$.

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Diophantine approximation by linear forms on manifolds

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Abstract. The following Khintchine-type theorem is proved for manifolds M embedded in $\mathbb{R}^k$ which satisfy some mild curvature conditions. The inequality $|\mathbf{q} \cdot \mathbf{x}| < \psi(|\mathbf{q}|)$ where $\psi(r) \rightarrow 0$ as $r \rightarrow \infty$ has finitely or infinitely many solutions $\mathbf{q} \in \mathbb{Z}^k$ for almost all (in induced measure) points $\mathbf{x}$ on M according as the sum $\sum_{r=1}^{\infty} \psi(r)r^{k-2}$ converges or diverges (the divergent case requires a slightly stronger curvature condition than the convergent case). Also, the Hausdorff dimension is obtained for the set (of induced measure 0) of point in M satisfying the inequality infinitely often when $\psi(r) = r^{-\tau}$, $\tau > k - 1$.

Keywords. Metric diophantine approximation; Khintchine's theorem; Hausdorff dimension; manifolds.

1. Introduction

In this paper we continue the discussion of the metric Diophantine approximation of points $\mathbf{x} = (x_1, \dots, x_k)$ lying on smooth submanifolds M of Euclidean space $\mathbb{R}^k$ which was begun in [4] and [5]. For manifolds M satisfying certain non-zero curvature conditions, the following Khintchine-type theorem was proved in [5]. Let $\psi(r) > 0$, $r = 1, 2, \dots$, tend to zero as $r \rightarrow \infty$, let $|\mathbf{x}| = \max \{|x_i| : i = 1, \dots, k\}$, and, for any $t \in \mathbb{R}$, let $\|t\| = \inf\{|t - p| : p \in \mathbb{Z}\}$. Then, for almost all $\mathbf{x} \in M$ (with respect to the induced Lebesgue measure on M), the inequality

$$\|\mathbf{q} \cdot \mathbf{x}\| < \psi(|\mathbf{q}|), \quad (1)$$

has infinitely many integer vector solutions $\mathbf{q} \in \mathbb{Z}^k$ if and only if the series $\sum_{r=1}^{\infty} \psi(r)r^{k-1}$ diverges. It has also been shown that the Hausdorff dimension of the set

$$\{\mathbf{x} \in M : \|\mathbf{q} \cdot \mathbf{x}\| < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^k\}, \quad (2)$$

is $m - 1 + (k + 1)/(\tau + 1)$, when $\tau > k$ (this follows readily from Theorem 1.3 of [4]).

In this paper we obtain analogous results for the stronger inequality

$$|\mathbf{q} \cdot \mathbf{x}| < \psi(|\mathbf{q}|). \quad (3)$$

Inequalities of this type arise as small divisors; for example the exceptional set in the KAM theorem consists of those points $\mathbf{x}$ for which, given any positive constant α and $\tau > k$, the linear form $\mathbf{q} \cdot \mathbf{x}$ satisfies $|\mathbf{q} \cdot \mathbf{x}| < \alpha |\mathbf{q}|^{-\tau}$ for some non-zero integer vector $\mathbf{q}$ (see [1], [7], [9]). This set is related to "lim-sup" sets of the form (2) with $M = \mathbb{R}^k$

we suppose that M is a C^3 manifold of dimension m and codimension n embedded in $\mathbb{R}^{m+n}$, where $m+n=k$. For any point $x \in M$, $T_x M$ and $T_x M^\perp$ will denote the tangent and normal spaces of M at x respectively; for any $\gamma \in T_x M^\perp$ we let $\kappa^i(x, \gamma)$, $i = 1, \dots, m$, denote the principal curvatures of M at x with respect to γ , see [4] and [8]. The following definitions are taken from [4] and [5].

DEFINITION

Let $x \in M$. We say that M satisfies *condition K0* at the point x , with respect to its embedding in $\mathbb{R}^{m+n}$, if for any $\gamma \in T_x M^\perp$ at least two of the principal curvatures $\kappa^i(x, \gamma)$ are non-zero. If, in addition, these curvatures have the same sign then *condition K1* holds at x . If there exists a codimension 1 hyperplane $N(x) \subset \mathbb{R}^{m+n}$, intersecting M transversely at x , such that $N(x) \cap M$ satisfies condition K1 at x with respect to its embedding in $N(x)$ then *condition K2* holds at x .

It is clear that if condition K1 holds at x then condition K0 must hold at x . In addition, if K2 holds at x then condition K1 holds at x , see [5].

We can now state our main results. Let

$$V(M, \tau) = \{x \in M : v(x) \geq \tau\}, \quad V'(M, \tau) = \{x \in M : v(x) = \tau\},$$

where, for any $x \in \mathbb{R}^{m+n}$, $v(x)$ denotes the supremum of the set of positive numbers v for which the inequality

$$|q \cdot x| < |q|^{-v}$$

has infinitely many solutions $q \in \mathbb{Z}^{m+n}$.

Theorem 1.1. *Suppose that condition K0 holds at all $x \in M$ except on a set of Hausdorff dimension at most $m-1$. Then*

$$\dim V(M, \tau) = \dim V'(M, \tau) = m-1 + (m+n)/(\tau+1),$$

for all $\tau > m+n-1$.

It can be verified that the dimension of the analogue of the set (2) for the inequality (3) is equal to the dimension of the sets $V(M, \tau)$ and $V'(M, \tau)$.

Theorem 1.2. *Suppose that condition K1 holds at almost all $x \in M$. If the series*

$$\sum_{r=1}^{\infty} \psi(r) r^{m+n-2} \tag{4}$$

is convergent then, for almost all $x \in M$, the inequality (3) has at most finitely many solutions $q \in \mathbb{Z}^{m+n}$.

Theorem 1.3. *Suppose that at almost all $x \in M$ condition K2 holds and, in addition, the hyperplane $N(x)$ can be chosen so that it contains one of the coordinate axes. If the series (4) is divergent then, for almost all $x \in M$, the inequality (3) has infinitely many solutions $q \in \mathbb{Z}^{m+n}$.*

We remark that from geometrical considerations it would seem natural to demand that the hyperplane $N(x)$ in the hypotheses of Theorem 1.3 should pass through the origin. The condition that it should also contain one of the coordinate axes arises from the method of proof and is probably not necessary for the truth of the theorem. However, this additional condition is not significantly more restrictive than the basic condition involving condition K2.

In the case that M is hypersurface (i.e. $n = 1$) the above results can be proved under a different, less restrictive, assumption.

DEFINITION

If M is a hypersurface and $\mathbf{x} \in M$, we say that M satisfies *condition K3* at $\mathbf{x}$ if every hyperplane H_0 in $\mathbb{R}^{m+n}$ passing through $\mathbf{x}$ and $\mathbf{0}$ is transverse to M at $\mathbf{x}$.

Condition K3 is not comparable with the conditions K_i , $i = 0, 1, 2$, in the sense that K3 can hold when the others do not, and vice versa. However, it is a rather weak condition and easily satisfied. In fact, only hypersurfaces which consist of portions of cones with vertex the origin can fail to satisfy it. Note also that condition K3 can hold for any dimension $m \geq 1$, unlike the earlier conditions, see [4]. However, if $n > 1$, then it is always possible to find a hyperplane H_0 through $\mathbf{0}$ tangential to M at any point $\mathbf{x} \in M$; thus condition K3 cannot hold if $n > 1$.

Theorem 1.4. *Suppose that M is a hypersurface. If condition K3 holds at all points $\mathbf{x} \in M$ except on a set of dimension at most $m - 1$, then the conclusion of Theorem 1.1 holds. If condition K3 holds at almost all $\mathbf{x} \in M$ then:*

- (i) *the conclusion of Theorem 1.2 holds;*
- (ii) *if either $m \leq 2$ and the sequence $\psi(r)r^{m-1}$, $r = 1, 2, \dots$, is monotonically decreasing, or $m \geq 3$, then the conclusion of Theorem 1.3 holds.*

We remark that the monotonicity conditions in Theorem 1.4, (ii), are similar to those imposed in Groshev's extension of Khintchine's theorem, which concerns the inequality (1) with $\mathbf{x} \in \mathbb{R}^k$, see [10].

2. Proofs of the theorems

We henceforth assume that M has the form

$$M = \{\mathbf{x} \in \mathbb{R}^{m+n} : \mathbf{x} = \xi(\mathbf{u}), \mathbf{u} \in \Omega\} = \xi(\Omega),$$

where Ω is a bounded open subset of $\mathbb{R}^m$ and the parametrization function $\xi: \Omega \rightarrow \mathbb{R}^{m+n}$ is C^3 and has the form $\xi(\mathbf{u}) = (u_1, \dots, u_m, \phi_1(\mathbf{u}), \dots, \phi_n(\mathbf{u})) = (\mathbf{u}, \phi(\mathbf{u}))$, $\mathbf{u} \in \Omega$, where $\phi: \Omega \rightarrow \mathbb{R}^n$ is C^3 . In addition, we may assume that the function ξ and its first three derivatives are uniformly bounded on Ω . These assumptions are not restrictive since it is always possible to cover M with a countable collection of sufficiently small local coordinate neighbourhoods on which the assumptions hold (after relabelling the coordinate axes if necessary), and it is sufficient to prove the theorems on any such neighbourhood. For simplicity we will call this coordinate neighbourhood M . Since the mapping $\xi: \Omega \rightarrow M$ maps sets of measure zero in Ω onto sets of measure zero on M and preserves Hausdorff dimension it is sufficient to prove the results on appropriate subsets of Ω rather than M .

Note that in the course of the proofs we may neglect subsets of M or Ω which are too small to affect the results of the theorems. Thus, in proving the dimensional results we may neglect sets having dimension less than $m - 1$, whereas for the measure results we may neglect sets of measure zero.

[5]. However, in the analysis of (3) we need only consider the sets $B(\mathbf{q}, 0; \psi(|\mathbf{q}|))$, $\mathbf{q} \in \mathbb{Z}^{m+n}$. Using these sets, and following the arguments in § 5 of [4] we can obtain the inequality

$$\dim V(M, \tau) \leq \dim V'(M, \tau) \leq m - 1 + (m + n)/(\tau + 1),$$

for any $\tau > m + n - 1$. To obtain the reverse inequality we also follow the method used in § 5 of [4]. In the course of the proof Minkowski's linear forms theorem (see Theorem III on p. 151 of [3]) is used to obtain the inequality (5.3) in [4]. To obtain a corresponding result in the present case it is necessary to suppose that u_m , say, is bounded away from zero for all $\mathbf{u} \in \Omega$. Letting S denote the set of points $\mathbf{u} \in \Omega$ for which $u_m = 0$, it is obvious that $\dim S \leq m - 1$, and hence the set S can be ignored in the proof. Since the set $\Omega - S$ can be decomposed into a countable union of open subsets on each of which u_m is bounded away from zero this assumption is not restrictive. Now, for any $N \geq 2$ and any $\mathbf{u} \in \Omega$, it follows from Minkowski's linear forms theorem that there exists a non-zero vector $\mathbf{q}(\mathbf{u}) \in \mathbb{Z}^{m+n}$ such that

$$|\mathbf{q}(\mathbf{u}) \cdot \xi(\mathbf{u})| \leq N^{-m-n+1} (\log N)^{2n},$$

$$|q_i(\mathbf{u})| \leq N, i = 1, \dots, m-1, \quad |q_i(\mathbf{u})| \leq N(\log N)^{-2}, i = m+1, \dots, m+n$$

(since $\xi_m(\mathbf{u}) = u_m$ is bounded away from zero on Ω). Since the function $\xi(\mathbf{u})$ is uniformly bounded on Ω and u_m is bounded away from zero, these inequalities imply that $|\mathbf{q}_m(\mathbf{u})| \leq c_1 N$, where $c_1 > 0$. These estimates provide the appropriate analogue of (5.3) in [4] for the present situation and enable us to complete the proof of Theorem 1.1 as in [4].

We now suppose that M satisfies condition K1 at the point $\mathbf{x}^0 = \xi(\mathbf{u}^0)$. To prove Theorem 1.2 we require the following lemmas. The first lemma is Lemma 3.1 of [4].

Lemma 2.1. Let J be an interval and let $f: J \rightarrow \mathbb{R}$ be a C^3 function and suppose that

$$\left| \frac{df}{du}(u) \right| \geq \alpha, u \in J, \quad (5)$$

for some $\alpha > 0$. Then, for any $\rho \geq 0$

$$|f^{-1}((-\rho, \rho))| \leq 2\alpha^{-1}\rho. \quad (6)$$

Lemma 2.2. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^2 function and suppose that

$$\left| \frac{\partial^2 f}{\partial u_i^2}(\mathbf{u}) \right| \geq \alpha, i = 1, 2, \quad \left| \frac{\partial^2 f}{\partial u_1 \partial u_2}(\mathbf{u}) \right| \leq \alpha/2, \mathbf{u} \in \mathbb{R}^2, \quad (7)$$

for some $\alpha > 0$. Then, for any $\rho \geq 0$.

$$|f^{-1}((-\rho, \rho))| \leq 32\pi\alpha^{-1}\rho. \quad (8)$$

Proof. It can be shown that the inequalities (7) imply that there exists a unique point $\mathbf{u}_{00} \in \mathbb{R}^2$ such that

$$\frac{\partial f}{\partial u_i}(\mathbf{u}_{00}) = 0, \quad i = 1, 2, \quad (9)$$

see §3 of [4]. We let $f_{00} = f(\mathbf{u}_{00})$. By translating the origin of the plane $\mathbb{R}^2$ we may assume that $\mathbf{u}_{00} = \mathbf{0}$. We now introduce polar coordinates $\mathbf{u}(r, \theta) = (r \cos \theta, r \sin \theta)$ and let $g(r, \theta) = f(\mathbf{u}(r, \theta))$. It follows easily from (7) and (9) that

$$\frac{\partial g}{\partial r}(0, \theta) = 0, \quad \frac{\partial^2 g}{\partial r^2}(r, \theta) \geq \alpha/2, \quad (10)$$

for all $r \geq 0, \theta \in [0, 2\pi]$. For any $\rho \geq 0$ and $\theta \in [0, 2\pi]$, let $G(\theta; \rho) = \{r \geq 0: |g(r, \theta)| < \rho\}$ and let

$$\eta(\theta; \rho) = \int_{G(\theta; \rho)} r dr.$$

Then

$$|f^{-1}((-\rho, \rho))| = \int_0^{2\pi} \eta(\theta; \rho) d\theta. \quad (11)$$

Letting $R(\theta; \rho) = \max \{r \geq 0: r \in G(\theta; \rho)\}$ it follows from (10) and Lemma 3.2 of [4] that

$$R(\theta; \rho) \leq (|f_{00}| + \rho)^{\frac{1}{2}} \alpha^{-\frac{1}{2}}, \quad |G(\theta; \rho)| \leq 8 \cdot 2^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \min(\rho |f_{00}|^{-\frac{1}{2}}, \rho^{\frac{1}{2}}),$$

(to apply Lemma 3.2 of [4] we extend the function $r \rightarrow g(r, \theta)$, for each θ , to the whole of $\mathbb{R}$ by reflection. By (10) this extension is C^2). Thus,

$$\begin{aligned} |\eta(\theta; \rho)| &\leq R(\theta; \rho) |G(\theta; \rho)| \\ &\leq \begin{cases} 2^{\frac{1}{2}} \rho^{\frac{1}{2}} \alpha^{-\frac{1}{2}} 8 \cdot 2^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \rho^{\frac{1}{2}} = 16 \alpha^{-1} \rho, & \text{if } |f_{00}| \leq \rho, \\ 2^{\frac{1}{2}} |f_{00}|^{\frac{1}{2}} \alpha^{-\frac{1}{2}} 8 \cdot 2^{1/2} \alpha^{-\frac{1}{2}} \rho |f_{00}|^{-\frac{1}{2}} = 16 \alpha^{-1} \rho, & \text{if } |f_{00}| \geq \rho, \end{cases} \end{aligned}$$

and so (8) follows from (11).

Lemma 2.3. *If Ω is sufficiently small then, for any $\rho \geq 0$ and $\mathbf{q} \in \mathbb{Z}^{m+n} \setminus \{\mathbf{0}\}$, $b \in \mathbb{Z}$,*

$$|B(\mathbf{q}, b; \rho)| \leq c_2 \rho |\mathbf{q}|^{-1},$$

where $c_2 > 0$ is a constant.

Proof. The proof is similar to the proof of Lemma 3.1 of [5], using Lemmas 2.1 and 2.2 above.

Theorem 1.2 now follows from the Borel–Cantelli lemma, together with the convergence of the series (4) and Lemma 2.3 (with $b = 0$, $\rho = \psi(|\mathbf{q}|)$).

Now suppose that the hypotheses of Theorem 1.3 hold at the point $\mathbf{x}^0 \in M$. In addition, we will suppose, for now, that the coordinate axis mentioned in the hypotheses is transverse to the tangent plane $T_{\mathbf{x}^0} M$ of M at $\mathbf{x}^0$. After relabelling the axes, if necessary, we may suppose that this axis is spanned by the vector $\mathbf{e}_{m+n} = (0, \dots, 0, 1) \in \mathbb{R}^{m+n}$. These assumptions imply that there is an orthogonal linear transformation of coordinates $\mathbf{x} \rightarrow \tilde{\mathbf{x}}$ in $\mathbb{R}^{m+n}$ with the following properties:

- (i) $x_{m+n} = \tilde{x}_{m+n}$;
- (ii) $N(\mathbf{x}^0) = \{\tilde{\mathbf{x}} \in \mathbb{R}^{m+n}: \tilde{x}_1 = 0\}$;
- (iii) in the $\tilde{\mathbf{x}}$ coordinates M has a parametrization $M = \boldsymbol{\mu}(\Lambda)$ near $\mathbf{x}^0$, where $\Lambda \subset \mathbb{R}^m$ is an open set and the function $\boldsymbol{\mu}: \Lambda \rightarrow \mathbb{R}^{m+n}$ has the form $\boldsymbol{\mu}(\mathbf{v}) = (\mathbf{v}, \lambda(\mathbf{v}))$, $\mathbf{v} \in \Lambda$.

We let $\mathbf{v}^0 = \boldsymbol{\mu}^{-1}(\tilde{\mathbf{x}}^0)$.

For any vectors $\mathbf{c} \in \mathbb{R}^m, \mathbf{d} \in \mathbb{R}^n$, let

$$\mu_{(\mathbf{c}, \mathbf{d})}(\mathbf{v}) = \mathbf{c} \cdot \mathbf{v} + \mathbf{d} \cdot \boldsymbol{\lambda}(\mathbf{v}), \quad \mathbf{v} \in \Lambda.$$

Since, in the $\tilde{\mathbf{x}}$ coordinates, $N(\mathbf{x}^0)$ is the plane $\tilde{x}_1 = 0$, condition K2 holds at $\tilde{\mathbf{x}}^0$ if and only if the intersection of M with this plane satisfies condition K1 at $\tilde{\mathbf{x}}^0$. A parametrization of the above form for this intersection is obtained from the function $\boldsymbol{\mu}$ by fixing $v_1 = 0$ and varying $\mathbf{v}' = (v_2, \dots, v_m)$. Using this parametrization we obtain the following equivalent condition for K2 to hold at $\tilde{\mathbf{x}}^0$:

K2' for all non-zero vectors $\mathbf{c}, \mathbf{d}$, the matrix $D'^2 \mu_{(\mathbf{c}, \mathbf{d})}(\mathbf{v}^0)$ has at least two non-zero eigenvalues of the same sign

(here D' denotes Frechet differentiation with respect to the variable $\mathbf{v}'$, i.e. keeping the first variable fixed). Note that the matrix $D'^2 \mu_{(\mathbf{c}, \mathbf{d})}(\mathbf{v}^0)$ is independent of $\mathbf{c}$, so for each non-zero $\mathbf{d}$ it is only necessary to check this condition for one vector $\mathbf{c}$. An immediate consequence of this condition is that

$$\text{rank } D'^2 \lambda_i(\mathbf{v}^0) \geq 2, i = 1, \dots, n. \quad (12)$$

The idea of the proof is to now divide (3) by the variable x_{m+n} to yield

$$|q_1 x_1 / x_{m+n} + \dots + q_{m+n-1} x_{m+n-1} / x_{m+n} + q_{m+n}| < \psi(|\mathbf{q}|) / x_{m+n}.$$

This transforms the inequality (1) to an inequality of the form (3) and we then use the results of [5]. A similar idea is used in [7], where the points $\mathbf{x}$ lie in Euclidean space, making the analysis much simpler. The main difficulty in the present setting, where the points $\mathbf{x}$ lie on the manifold M , is that the transformed points $(x_1 / x_{m+n}, \dots, x_{m+n-1} / x_{m+n})$ lie on another manifold N , say, and it is necessary to ensure that the appropriate curvature conditions are satisfied by N . This causes several technical complications.

It follows from (12) that λ_n is non-zero except on a set of measure zero. Thus, without loss of generality, we may suppose that $\lambda_n(\mathbf{v}^0) \neq 0$, and hence, if Λ is sufficiently small, $|\lambda_n(\mathbf{v})| > \delta > 0, \mathbf{v} \in \Lambda$. We now define a transformation $T: \Lambda \rightarrow \mathbb{R}^m$ by

$$T(\mathbf{v}) = \boldsymbol{\tau} / \lambda_n(\mathbf{v}), \quad \mathbf{v} \in \Lambda.$$

Lemma 2.4. *If Λ is sufficiently small then, for almost all $\mathbf{v} \in \Lambda$, the linear mapping $DT(\mathbf{v}): \mathbb{R}^m \rightarrow \mathbb{R}^m$ is non-singular.*

Proof. By direct calculation it can be verified that $DT(\mathbf{v})$ is singular if and only if $\lambda_n(\mathbf{v})$ is an eigenvalue of the matrix $(v_i \partial \lambda_n(\mathbf{v}) / \partial v_j)_{i,j=1}^m$. The only eigenvalue of this matrix is $\mathbf{v} \cdot D \lambda_n(\mathbf{v})$, so $DT(\mathbf{v})$ is singular only when

$$\mathbf{v} \cdot D \lambda_n(\mathbf{v}) - \lambda_n(\mathbf{v}) = 0. \quad (13)$$

Using (12) it can be seen that near $\mathbf{v}^0$ this condition holds only on a set of measure zero.

By Lemma 2.4 we may suppose, without loss of generality, that $DT(\mathbf{v}^0)$ is non-singular, and hence, if Λ is sufficiently small, that $T: \Lambda \rightarrow \mathbb{R}^m$ is a C^3 diffeomorphism. Let $\Gamma = T(\Lambda)$. Clearly Γ is an open set. Now let $\hat{\boldsymbol{\lambda}}(\mathbf{v}) = (\lambda_1(\mathbf{v}), \dots, \lambda_{n-1}(\mathbf{v}))$, $\mathbf{v} \in \Lambda$. We

define the manifold $N \subset \mathbb{R}^{m+n-1}$ by the representation

$$N = \{\mathbf{y} \in \mathbb{R}^{m+n-1} : \mathbf{y} = (\mathbf{v}, \hat{\lambda}(\mathbf{v}))/\lambda_n(\mathbf{v}), \mathbf{v} \in \Lambda\}. \quad (14)$$

Writing $\mathbf{v} = T^{-1}(\mathbf{w})$, $\mathbf{w} \in \Gamma$, and $\boldsymbol{\eta}(\mathbf{w}) = \hat{\lambda}(\mathbf{v})/\lambda_n(\mathbf{v})$, $\boldsymbol{\zeta}(\mathbf{w}) = (\mathbf{w}, \boldsymbol{\eta}(\mathbf{w}))$, then $N = \boldsymbol{\zeta}(\Gamma)$. We let $\mathbf{w}^0 = T(\mathbf{v}^0)$ and $\mathbf{y}^0 = \boldsymbol{\zeta}(\mathbf{w}^0) \in N$.

Lemma 2.5. The manifold N satisfies condition K2 at the point $\mathbf{y}^0$. The corresponding transverse hyperplane can be taken to be the plane $\{\mathbf{y} \in \mathbb{R}^{m+n-1} : y_1 = 0\}$.

Proof. For any $\mathbf{c} \in \mathbb{R}^m$, $\hat{\mathbf{d}} \in \mathbb{R}^{n-1}$, let

$$\zeta_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{w}) = \mathbf{c} \cdot \mathbf{w} + \hat{\mathbf{d}} \cdot \boldsymbol{\eta}(\mathbf{w}), \mathbf{w} \in \Lambda.$$

By the above characterization K2' of condition K2, to prove the lemma it suffices to show that for each non-zero $\hat{\mathbf{d}} \in \mathbb{R}^{n-1}$ the matrix $D'^2 \zeta_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{w}^0)$ has two non-zero eigenvalues of the same sign. For $\mathbf{v} \in \Lambda$ let

$$G_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}) = \mathbf{c} \cdot \mathbf{v} + \hat{\mathbf{d}} \cdot \hat{\lambda}(\mathbf{v}).$$

$$H_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}) = G_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v})/\lambda_n(\mathbf{v}).$$

Clearly we have

$$\zeta_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{w}) = H_{(\mathbf{c}, \hat{\mathbf{d}})}(T^{-1}(\mathbf{w})). \quad (15)$$

Now choose some non-zero $\hat{\mathbf{d}} \in \mathbb{R}^{n-1}$ and consider this fixed. Since $DT(\mathbf{v}^0)$ is non-singular we can choose $\mathbf{c} \in \mathbb{R}^m$ such that

$$DH_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0) = 0. \quad (16)$$

By differentiating (15) and using (16) we find that

$$D'^2 \zeta_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{w}^0) = [D' T(\mathbf{v}^0)^{-1}]^t D'^2 H_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0) D' T(\mathbf{v}^0)^{-1}, \quad (17)$$

where the superscript t denotes transpose. It can be shown that the relationship (17) implies that the matrices $D'^2 \zeta_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{w}^0)$ and $D'^2 H_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0)$ have the same number of positive, negative and zero eigenvalues. Now using (16) again, it can be shown that

$$\begin{aligned} D'^2 H_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0) &= \lambda_n(\mathbf{v}^0)^{-1} D'^2 G_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0) - G_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0) D'^2 \lambda_n(\mathbf{v}^0) \\ &= D'^2 \mu_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0), \end{aligned}$$

where $\mathbf{g} \in \mathbb{R}^n$ is the non-zero vector

$$\mathbf{g} = (\lambda_n(\mathbf{v}^0)^{-1} \hat{\mathbf{d}}, G_{(\mathbf{c}, \hat{\mathbf{d}})}(\mathbf{v}^0)).$$

Whence the lemma follows from the fact that M satisfies K2' at $\mathbf{x}^0$.

Now consider the inequality

$$|\mathbf{q} \cdot \boldsymbol{\xi}(\mathbf{u})| < \psi(|\mathbf{q}|), \quad (18)$$

for some $\mathbf{v} \in \Lambda$. Since $\tilde{x}_{m+n} = x_{m+n}$ for all $\mathbf{x}$, we have $\tilde{\mathbf{q}} = (q_1, \dots, \tilde{q}_{m+n-1}, q_{m+n})$, for any $\mathbf{q} \in \mathbb{Z}^{m+n}$. We write $\hat{\mathbf{q}} = (\tilde{q}_1, \dots, \tilde{q}_{m+n-1})$, and let $\hat{Q}$ denote the set of such vectors. By definition $\hat{Q}$ is obtained from the lattice of integer vectors $(q_1, \dots, q_{m+n-1}) \in \mathbb{Z}^{m+n-1}$ by an orthogonal transformation. For any $\hat{\mathbf{q}} \in \hat{Q}$ let $[\hat{\mathbf{q}}] = |(q_1, \dots, q_{m+n-1})|$, where $(q_1, \dots, q_{m+n-1})$ is the corresponding element of $\mathbb{Z}^{m+n-1}$. Dividing (19) by $\lambda_n(\mathbf{v})$ and using the above definitions yields the inequality

$$|\hat{\mathbf{q}} \cdot \boldsymbol{\zeta}(\mathbf{w}) + q_{m+n}| < \psi(|\mathbf{q}|)/\lambda_n(\mathbf{v}), \quad (20)$$

where $\mathbf{w} = T(\mathbf{v})$ while if $|q_{m+n}| \leq [\hat{\mathbf{q}}]$, then $[\hat{\mathbf{q}}] = |\mathbf{q}|$ and we obtain

$$|\hat{\mathbf{q}} \cdot \boldsymbol{\zeta}(\mathbf{w}) + q_{m+n}| < \psi([\hat{\mathbf{q}}])/\lambda_n(\mathbf{v}). \quad (21)$$

The inequality (21) is similar to the inequality (1), with $\mathbf{x} = \boldsymbol{\zeta}(\mathbf{w})$ lying on the manifold N , which is discussed in [5]. The only differences are that $\hat{\mathbf{q}}$ belongs to the lattice $\hat{Q}$ rather than the lattice of integer vectors, and the right hand side of (21) depends on $[\hat{\mathbf{q}}]$ rather than $|\hat{\mathbf{q}}|$ and contains the factor $1/\lambda_n(\mathbf{v})$. However, these differences are not crucial ($\lambda_n(\mathbf{v})$ is bounded away from zero on Λ) and, since N satisfies condition K2 at $\mathbf{y}^0$, the methods of §4 of [5] can be used to show that if Γ is sufficiently small then, for almost all $\mathbf{w} \in \Gamma$, (21) has infinitely many solutions $\hat{\mathbf{q}} \in \hat{Q}$, $q_{m+n} \in \mathbb{Z}$, with $|q_{m+n}| \leq [\hat{\mathbf{q}}]$. Since T is a diffeomorphism and maps sets of measure zero onto sets of measure zero, it follows from the above discussion that (18) has infinitely many solutions $\mathbf{q}$ for almost all $\mathbf{u} \in \Omega$, which proves Theorem 1.3 under the above transversality assumption.

We now consider the case where the coordinate axis referred to in the hypotheses of Theorem 1.3 is not transverse to $T_{\mathbf{x}^0}M$. In this case we will suppose (by relabelling the axes if necessary) that this axis is spanned by the vector $\mathbf{e}_m = (0, \dots, 1, \dots, 0)$ (1 in the m th position). A coordinate transformation $\mathbf{x} \rightarrow \tilde{\mathbf{x}}$ again exists with the properties (ii) and (iii) above, but with (i) replaced by: (i)' $x_m = \tilde{x}_m$. The proof of the theorem now proceeds in a similar manner to the above except that we divide by the variable v_m rather than by $\lambda_n(\mathbf{v})$ in the above constructions. This does not entail any changes in the strategy of the proof, but the details have to be rechecked. For brevity we omit this. This completes the proof of Theorem 1.3.

Now suppose that the hypotheses of Theorem 1.4 hold at the point $\mathbf{x}^0 \in M$. The first part of the theorem can be proved as in [4]. In fact, the proof of the main estimate, Lemma 4.1 of [4], is rather simpler in the present situation since the transversality assumptions in condition K3 ensure that only the second case in the proof can occur here. Part (i) of the theorem can be proved in the same manner as Theorem 1.2 above, using Lemma 2.3, which in this case can easily be proved directly using the transversality assumptions in condition K3.

To prove part (ii) we note that it can be shown that condition K3 holds at any point $\mathbf{x}^0 = \boldsymbol{\xi}(\mathbf{u}^0)$ if and only if

$$\mathbf{u}^0 \cdot D\phi_1(\mathbf{u}^0) - \phi_1(\mathbf{u}^0) \neq 0 \quad (22)$$

(recall that $n = 1$ here, so $\boldsymbol{\xi}(\mathbf{u}) = (\mathbf{u}, \phi_1(\mathbf{u}))$, $\mathbf{u} \in \Omega$). Since condition K3 holds almost

everywhere it follows from (22) that $\phi_1(\mathbf{u}) \neq 0$ for almost all $\mathbf{u} \in \Omega$. Thus, without loss of generality, we may choose $\mathbf{u}^0 \in \Omega$ such that (22) holds and $\phi_1(\mathbf{u}) \neq 0$ for all $\mathbf{u}$ in a sufficiently small open neighbourhood Ω of $\mathbf{u}^0$. We define a transformation $T: \Omega \rightarrow \mathbb{R}^m$ by $T(\mathbf{u}) = \mathbf{u}/\phi_1(\mathbf{u})$, $\mathbf{u} \in \Omega$. It is shown in the proof of Lemma 2.4 that (22) implies that $DT(\mathbf{u}^0)$ is non-singular and so, for sufficiently small Ω , the mapping T is a C^3 diffeomorphism. We let $\mathbf{w}^0 = T(\mathbf{u}^0)$ and $\Gamma = T(\Omega)$. Now, dividing the inequality (18) by $\phi_1(\mathbf{u})$ yields

$$|\hat{\mathbf{q}} \cdot T(\mathbf{u}) + q_{m+1}| < \psi(|\mathbf{q}|)/\phi_1(\mathbf{u}). \quad (23)$$

Thus, we can prove the result using similar arguments to those used to prove Theorem 1.3 above. Note, however, that $T(\mathbf{u})$ varies over an open set $\Gamma \subset \mathbb{R}^m$ as $\mathbf{u}$ varies over Ω , so we may use the theory of Diophantine approximation in Euclidean space, see [10], to give the required results for the inequality (23). This contrasts with the proof of Theorem 1.3 where it was necessary to consider the approximation of points on the manifold N . The more complete theory of approximation in Euclidean space enables us to deal with the cases $m = 1$ and $m = 2$, which could not be tackled in Theorem 1.3. This completes the proof of Theorem 1.4.

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Lebesgue-Stieltjes integral inequalities in several variables with retardation

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Abstract. The aim of this paper is to establish some new Lebesgue-Stieltjes integral inequalities in n independent variables with retardation which generalize and unify continuous and discrete inequalities of the Gronwall–Bellman–Bihari type in n independent variables.

Keywords. Gronwall inequality; Lebesgue-Stieltjes integral; retardation.

1. Introduction

One of the most useful techniques used in the theory of ordinary or stochastic differential equations and integral equations is applying the celebrated Gronwall–Bellman–Bihari inequality. Gronwall inequality appeared in 1919 in [8]. In 1961, the book “Inequalities” by Beckenbach and Bellman [3] contained a number of results on inequalities. Some new results can be found in Bihari [4], Donnan [5], Dhongade and Deo [6–7], Pachpatte [10–11], Raghavendra and Ras [13]. In recent years, more than one variable generalizations of this inequality have evoked considerable interest since they are handy tools. Particularly we refer to the results of Agarwal [1], Akinyele [2], Pachpatte [12], Singare and Pachpatte [14–15], Thandapani and Pachpatte [16], Yeh [17, 19–21], Yeh and Shih [18], Young [22], Zahariew and Bainov [23] among others.

Recently, we established some new Lebesgue-Stieltjes integral inequalities in several variables [9] which extended and unified the continuous and discrete inequalities of Gronwall–Bellman–Bihari type obtained by the above authors. In this paper we will employ the two essential inequalities in our previous paper [7] (see the following Lemmas 2.1 and 2.2) to establish some new Lebesgue-Stieltjes integral inequalities in n independent variables with retardation which extend some recent results in Akinyele [2], Donnan [5] and others mentioned above. We then use these results to deal with discrete inequalities of the Gronwall–Bellman–Bihari type in n independent variables with retardation which generalize some recent results of Singare and Pachpatte [14–15], Yeh [20–21].

2. Notations and Lemmas

Unless specified otherwise we will adopt the following notations for the remainder of the paper. The origin of n -dimensional Euclidean space R^n is denoted by

$0 = (0, \dots, 0)$. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R^n, x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, \dots, n$. Let $X = (X_1, \dots, X_n) \in R^n$ be a fixed point with $X \geq 0$. Let $[0, X] = [0, X_1] \times \dots \times [0, X_n]$. Let $A(t) = (A_1(t), \dots, A_n(t))$, where $A_i(t): [0, X_i] \rightarrow R_+ (= [0, \infty))$ is right continuous nondecreasing function with $A_i(0) = 0$ for all $i = 1, \dots, n$. Let $x = (x_1, \dots, x_n)$, denote

$$\int_0^{x_1} \dots \int_0^{x_n} \dots dA_n(s_n) \dots dA_1(s_1)$$

by

$$\int_0^x \dots dA(s).$$

A function $f: [0, X] \rightarrow R$ is said to be left limit if there exists $\lim_{s \uparrow x} f(s)$ for all $0 < x \leq X$. In this case we will denote $\lim_{s \uparrow x} f(s)$ by $f(s-)$, here we let $f(0-) = f(0)$. Particularly, we will use the notation

$$\int_0^{x-} \dots dA(s) = \lim_{y \uparrow x} \int_0^y \dots dA(s).$$

Denote by $\mathcal{M}([0, X], R_+)$ the family of all Borel measurable, bounded, left limit and nonnegative functions defined on $[0, X]$. Denote by $\mathcal{B}([0, X], R_+)$ the family of all bounded, Borel measurable and nonnegative functions defined on $[0, X]$. We also denote by $\mathcal{F}$ the class of all Borel measurable function $\sigma: [0, X] \rightarrow [0, X]$ such that $\sigma(x) \leq x$ for all $0 \leq x \leq X$.

Let $\mathcal{H}$ stand for the class of all function $H: R_+ \rightarrow R_+$ satisfying

- (1) $H(v)$ is nondecreasing and continuous for $v \geq 0$ and positive for $v > 0$;
- (2) there exists a continuous function Φ on R_+ such that $H(\alpha v) \leq \Phi(\alpha)H(v)$ for all $\alpha > 0$ and $v \geq 0$.

In this case H is said to belong to class $\mathcal{H}$ with corresponding multiplier function Φ . It is obvious that H belongs to class $\mathcal{H}$ with itself as the corresponding multiplier function if H satisfies condition (1) and is submultiplicative.

We also denote by $\mathcal{K}$ all of function $K \in \mathcal{H}$ with corresponding multiplier function Φ and, furthermore, addition function φ which is continuous on R_+ such that $K(\alpha + v) \leq \varphi(\alpha) + K(v)$ for all $\alpha \geq 0$ and $v \geq 0$. Clearly, if K satisfies a Lipschitz condition

$$|K(\alpha + v) - K(v)| \leq k\alpha, \quad \alpha, v \geq 0,$$

then we can choose $\varphi(\alpha) = k\alpha$.

For the readers convenience we state the following lemmas due to the author [9, Corollary 3.3 and Theorem 3.1] which will be used in the remainder of this paper.

Lemma 2.1. Let $u(x) \in \mathcal{M}([0, X], R_+)$, $f(s) \in \mathcal{B}([0, X], R_+)$ and $u_0 \geq 0$. Then the inequality

$$u(x) \leq u_0 + \int_0^x f(s)u(s-)dA(s), \quad 0 \leq x \leq X, \quad (1)$$

Lemma 2.2. Let $u(x) \in \mathcal{M}([0, X], R_+)$, $f(x) \in \mathcal{B}([0, X], R_+)$ and $u_0 > 0$. Let H be a continuous function defined on $[0, \infty)$ into $[0, \infty)$ such that $H(v)$ is positive and nondecreasing for $v > 0$. Then inequality

$$u(x) \leq u_0 + \int_0^x f(s)H(u(s-))dA(s), \quad 0 \leq x \leq X, \quad (3)$$

implies

$$u(x) \leq G^{-1}\left(G(u_0) + \int_0^x f(s)dA(s)\right), \quad 0 \leq x < \hat{x}, \quad (4)$$

where

$$G(r) = \int_\varepsilon^r (1/H(v))dv, \quad r > 0, \varepsilon > 0, \quad (5)$$

G^{-1} is the inverse of G , and $\hat{x}$ is chosen so that

$$G(u_0) + \int_0^x f(s)dA(s) \in \text{Dom}(G^{-1}), \quad 0 \leq x \leq \hat{x}.$$

If $H(0) > 0$, we can define

$$G(r) = \int_0^r (1/H(v))dv, \quad r \geq 0. \quad (6)$$

In this case we even allow $u_0 \geq 0$.

It should be pointed out that all the results in this paper except §4 still hold if the origin 0 of the n -dimensional Euclidean space R^n is replaced by arbitrary $x_0 \in R^n$ with $x_0 \leq X$.

3. Main results

We begin with the following theorem.

Theorem 3.1. Assume $u(x), q(x) \in \mathcal{M}([0, X], R_+)$ with $q(x) \geq 1$, $f(x), g(x) \in \mathcal{B}([0, X], R_+)$, $\sigma(x) \in \mathcal{F}$, $H \in \mathcal{H}$ with corresponding multiplier function Φ , and $n(x)$ is a positive and nondecreasing function on $[0, X]$. Suppose

$$\begin{aligned} u(x) \leq n(x) + q(x) \left\{ \int_0^x f(s)H \left[u(\sigma(s)-) + q(\sigma(s)-) \right. \right. \\ \left. \left. \times \int_0^{\sigma(s)-} g(\tau)H(u(\sigma(\tau)-))dA(\tau) \right] dA(s) \right\} \end{aligned} \quad (7)$$

holds for all $0 \leq x \leq X$. Then

$$u(x) \leq n(x)q(x)G^{-1}\left(G(1) + \int_0^x [\{f(s) + g(s)\}/n(s)]\right)$$

for all $0 \leq x \leq \hat{x}$, where G and G^{-1} are defined as in Lemma 2.2, $\hat{x}$ is chosen so that

$$G(1) + \int_0^x [\{f(s) + g(s)\}/n(s)] \Phi(q(\sigma(s) -)n(\sigma(s) -)) dA(s) \\ \in \text{Dom}(G^{-1}), \quad 0 \leq x \leq \hat{x}.$$

Proof. Set

$$w(x) = u(x) + q(x) \int_0^x g(s) H(u(\sigma(s) -)) dA(s), \quad 0 \leq x \leq X.$$

Then

$$u(x) \leq w(x)$$

and

$$u(x) \leq n(x) + q(x) \int_0^x f(s) H(w(\sigma(s) -)) dA(s).$$

Therefore

$$w(x) \leq n(x) + q(x) \int_0^x (f(s) + g(s)) H(w(\sigma(s) -)) dA(s).$$

Thus

$$w(x) \leq n(x) q(x) \left(1 + \int_0^x [(f(s) + g(s))/n(s)] H(w(\sigma(s) -)) dA(s) \right), \quad 0 \leq x \leq X. \quad (9)$$

Defining

$$z(x) = 1 + \int_0^x [(f(s) + g(s))/n(s)] H(w(\sigma(s) -)) dA(s), \quad (10)$$

we have

$$w(x) \leq n(x) q(x) z(x). \quad (11)$$

Consequently

$$w(\sigma(s) -) \leq n(\sigma(s) -) q(\sigma(s) -) z(\sigma(s) -) \leq n(\sigma(s) -) q(\sigma(s) -) z(s -).$$

Therefore, from (10)

$$z(x) \leq 1 + \int_0^x [(f(s) + g(s))/n(s)] \Phi(n(\sigma(s) -) q(\sigma(s) -)) H(z(s -)) dA(s).$$

Applying Lemma 3.2 to this and using (11) we deduce desired result (8) immediately. The theorem has been proved.

Remark 3.1. If $A(x) = x$, $g(x) = 0$, and all the other functions in Theorem 3.1 are continuous and H is also submultiplicative, then Theorem 3.1 reduces to Theorem 4 of Akinyele [2]. In one-dimensional case, Theorem 3.1 gives a generalization of Dannan [5, Theorem 1].

Theorem 3.2. Let $u(x), f(x), g(x) \in \mathcal{M}([0, X], R_+)$, $h(x), k(x) \in \mathcal{B}([0, X], R_+)$, $\sigma(x) \in \mathcal{F}$,

and $K \in \mathcal{K}$ with corresponding multiplier function Φ and addition function φ . If

$$u(x) \leq f(x) + g(x) \left\{ \int_0^x h(s) K \left[u(\sigma(s) -) + g(\sigma(s) -) \right. \right. \\ \left. \left. \times \int_0^{\sigma(s)-} k(\tau) K(u(\sigma(\tau) -)) dA(\tau) \right] dA(s) \right\} \quad (12)$$

holds for all $0 \leq x \leq X$, then for $0 \leq x \leq \hat{x}$, we have

$$u(x) \leq f(x) + g(x) F^{-1} \left\{ F \left(\int_0^x (h(s) + k(s)) \varphi(f(\sigma(s) -)) dA(s) \right) \right. \\ \left. + \int_0^x (h(s) + k(s)) \Phi(g(\sigma(s) -)) dA(s) \right\} \quad (13)$$

for all $0 \leq x \leq \hat{x}$, where

$$F(r) = \int_\varepsilon^r (1/K(v)) dv, \quad r > 0, \varepsilon > 0, \quad (14)$$

F^{-1} is the inverse function of F , and $\hat{x}$ is chosen so that

$$F \left(\int_0^x (h(s) + k(s)) \varphi(f(\sigma(s) -)) dA(s) \right) \\ + \int_0^x (h(s) + k(s)) \Phi(g(\sigma(s) -)) dA(s) \in \text{Dom}(F^{-1})$$

for all $0 \leq x \leq \hat{x}$.

Proof. Set

$$w(x) = u(x) + g(x) \int_0^x k(s) K(u(\sigma(s) -)) dA(s), \quad 0 \leq x \leq X. \quad (15)$$

Then $u(x) \leq w(x)$ and

$$w(x) \leq u(x) + g(x) \int_0^x k(s) K(w(\sigma(s) -)) dA(s).$$

Putting (12) into this we deduce

$$w(x) \leq f(x) + g(x) \int_0^x (h(s) + k(s)) K(w(\sigma(s) -)) dA(s). \quad (16)$$

Define

$$z(x) = \int_0^x (h(s) + k(s)) K(w(\sigma(s) -)) dA(s), \quad 0 \leq x \leq X. \quad (17)$$

Thus

$$u(x) \leq w(x) \leq f(x) + g(x) z(x). \quad (18)$$

and from (17)

$$\begin{aligned} z(x) &\leq \int_0^x (h(s) + k(s))K(f(\sigma(s)-) + g(\sigma(s)-)z(\sigma(s)-))dA(s) \\ &\leq \int_0^x (h(s) + k(s))\varphi(f(\sigma(s)-))dA(s) \\ &\quad + \int_0^x (h(s) + k(s))\Phi(g(\sigma(s)-))K(z(\sigma(s)-))dA(s). \end{aligned}$$

By Lemma 2.2 and (18) we deduce desired result (13) and the proof is complete.

Theorem 3.3. Let $u(x)$, $f(x)$, $h(x)$ and $\sigma(x)$ be defined as in Theorem 3.2 with $f(x)$ nondecreasing and $u_0 \geq 0$. Let $Q(v) \in C(R_+, R_+)$ be nondecreasing with $r^{-1}Q(v) \leq Q(r^{-1}v)$ for all $r \geq 1$ and $v \geq 0$. Let $P(v) \in C(R_+, [1, \infty))$ be a strictly increasing, subadditive and supermultiplicative function. If the inequality

$$u(x) \leq f(x) + P^{-1}\left(Q\left\{u_0 + \int_0^x h(s)P(u(\sigma(s)-))dA(s)\right\}\right), \quad 0 \leq x \leq X \quad (19)$$

holds, where P^{-1} is the inverse function of P , then we have

$$\begin{aligned} u(x) &\leq f(x)P^{-1}\left\{1 + Q\left[U^{-1}\left(U(u_0) + \int_0^x [h(s)/P(f(s))]\right.\right.\right. \\ &\quad \left.\left.\left.\times P(f(\sigma(s)-))dA(s)\right)\right]\right\} \end{aligned} \quad (20)$$

for all $0 \leq x \leq \hat{x}$, where

$$U(r) = \int_0^r [1/(1 + Q(v))]dv, \quad r \geq 0, \quad (21)$$

U^{-1} is the inverse function of U , and $\hat{x}$ is chosen so that

$$U(u_0) + \int_0^x [h(s)/P(f(s))]P(f(\sigma(s)-))dA(s) \in \text{Dom}(U^{-1}), \quad 0 \leq x \leq \hat{x}.$$

Proof. Since P is subadditive, $P(f(x)) \geq 1$ is nondecreasing and $r^{-1}Q(v) \leq Q(r^{-1}v)$ for all $r \geq 1$ and $v \geq 0$, it follows from (19) that

$$P(u(x)) \leq P(f(x))\left(1 + Q\left\{u_0 + \int_0^x [h(s)/P(f(s))]P(u(\sigma(s)-))dA(s)\right\}\right). \quad (22)$$

Set

$$w(x) = u_0 + \int_0^x [h(s)/P(f(s))]P(u(\sigma(s)-))dA(s)$$

we then have

and

$$w(x) \leq u_0 + \int_0^x [h(s)/P(f(s))] P(f(\sigma(s)-)) (1 + Q(w(s-))) dA(s).$$

An application of Lemma 2.2 to this yields

$$w(s) \leq U^{-1} \left(U(u_0) + \int_0^x [h(s)/P(f(s))] P(f(\sigma(s)-)) dA(s) \right), \quad 0 \leq x \leq \hat{x}. \quad (24)$$

Since P is supermultiplicative and increasing, P^{-1} is submultiplicative. Thus, from (23)

$$u(x) \leq f(x) P^{-1}(1 + Q(w(x))).$$

Putting (24) into this we get the desired result (20).

We now let $B(x) = (B_1(t), \dots, B_n(t))$, where $B_i(t): [0, X_i] \rightarrow R_+$ is a right continuous nondecreasing function with $B_i(0) = 0$ for all $i = 1, \dots, n$. Similarly, we also use notation

$$\int_0^x \dots dB(s) = \int_0^{x_1} \dots \int_0^{x_n} \dots dB_n(s_n) \dots dB_1(s_1).$$

The following theorem generalizes Theorem 2 of Dannan [5].

Theorem 3.4. Let $f(x), g(x) \in \mathcal{B}([0, X], R_+)$, $\sigma(x) \in \mathcal{T}$, $u_0 > 0$, $u(x), q(x) \in \mathcal{M}([0, X], R_+)$ with $q(x) \geq 1$ and nondecreasing. Let $H \in \mathcal{H}$ with corresponding multiplier function Φ . If inequality

$$\begin{aligned} u(x) \leq & u_0 + q(x) \int_0^x f(s) H(u(\sigma(s)-)) dA(s) \\ & + \int_0^x g(s) \left\{ \int_0^{s-} f(\tau) H(u(\sigma(\tau)-)) dA(\tau) \right\} dB(s). \end{aligned} \quad (25)$$

holds for all $0 \leq x \leq X$, we then have

$$u(x) \leq q(x) E(x) G^{-1} \left[G(u_0) + \int_0^x f(s) \Phi(q(\sigma(s)-) E(\sigma(s)-)) dA(s) \right], \quad (26)$$

$$0 \leq x \leq \hat{x},$$

where

$$E(x) = \exp \left\{ \int_0^x [g(s)/q(s-)] dB(s) \right\},$$

G and G^{-1} are defined as in Lemma 2.2, $\hat{x}$ is such that

$$G(u_0) + \int_0^x f(s) \Phi(E(\sigma(s)-) q(\sigma(s)-)) dA(s) \in \text{Dom}(G^{-1}), \quad 0 \leq x \leq \hat{x}.$$

Proof. Define, for $0 \leq x \leq X$,

$$w(x) = \int_0^x f(s) H(u(\sigma(s)-)) dA(s) \quad (27)$$

and

$$z(x) = u_0 + q(x)w(x) + \int_0^x g(s)w(s-)dB(s). \quad (28)$$

Thus, from (25) and (28) we have

$$u(x) \leq z(x) \text{ and } w(x) \leq z(x)/q(x). \quad (29)$$

Hence, it follows from (28) that

$$z(x) \leq u_0 + q(x)w(x) + \int_0^x [g(s)/q(s-)]z(s-)dB(s). \quad (30)$$

Since $u_0 + q(x)w(x)$ is nondecreasing, we can apply Lemma 2.1 to (30) to get

$$z(x) \leq (u_0 + q(x)w(x))E(x). \quad (31)$$

Putting (27) into this and noting (29) and $q(x) \geq 1$ we deduce

$$z(x) \leq q(x)E(x) \left\{ u_0 + \int_0^x f(s)H(z(\sigma(s)-))dA(s) \right\}. \quad (32)$$

Define

$$J(x) = u_0 + \int_0^x f(s)H(z(\sigma(s)-))dA(s),$$

we then have from (32)

$$z(x) \leq q(x)E(x)J(x). \quad (33)$$

Hence

$$J(x) \leq u_0 + \int_0^x f(s)\Phi(q(\sigma(s)-)E(\sigma(s)-))H(J(s-))dA(s).$$

By Lemma 2.2, this, together with (33) and (29), yields desired result (26) and the proof is complete.

In the above theorems, the nonlinear function H or K is assumed to belong to certain class of functions. In what follows we assume H is Lipschitz continuous.

Theorem 3.5. Let $f(x), g(x) \in \mathcal{B}([0, X], R_+)$, $u(x), q(x), n(x) \in \mathcal{M}([0, X], R_+)$. Let $H: R_+ \rightarrow R_+$ be a nondecreasing function satisfying a Lipschitz condition

$$|H(v+r) - H(v)| \leq kr, \quad v, r \geq 0,$$

where k is a positive constant. Suppose

$$\begin{aligned} u(x) \leq n(x) + q(x) & \left[\int_0^x f(s)H(u(s-))dA(s) \right. \\ & \left. + \int_0^x g(s) \left\{ \int_0^{s-} f(\tau)H(u(\tau-))dA(\tau) \right\} dA(s) \right] \end{aligned} \quad (34)$$

holds for all $0 \leq x \leq X$. Then

$$w(x) \leq n(x) + q(x) \int_0^x f(s)H(n(s-))dA(s) \exp \left\{ \int_0^x [kf(s)q(s-) + g(s)]dA(s) \right\} \quad (35)$$

for all $0 \leq x \leq X$.

Proof. For $0 \leq x \leq X$, set

$$v(x) = \int_0^x f(s)H(u(s-))dA(s) \quad (36)$$

and

$$w(x) = v(x) + \int_0^x g(s)v(s-)dA(s). \quad (37)$$

Then

$$u(x) \leq n(x) + q(x)w(x) \text{ and } v(x) \leq w(x). \quad (38)$$

Putting (36) into (37) and noticing (38) we get

$$\begin{aligned} w(x) &\leq \int_0^x f(s)H(n(s-))dA(s) + \int_0^x g(s)w(s-)dA(s) \\ &\leq \int_0^x f(s)H(n(s-))dA(s) + \int_0^x [kf(s)q(s-) + g(s)]w(s-)dA(s). \end{aligned}$$

By Lemma 2.1, we have

$$w(x) \leq \int_0^x f(s)H(n(s-))dA(s) \exp \left\{ \int_0^x [kf(s)q(s-) + g(s)]dA(s) \right\}.$$

Putting this into (38) we get the desired result (35). The theorem has been proved.

4. Discrete inequalities

The aim of this section is to study discrete inequalities of the Gronwall–Bellman–Bihari type in n independent variables. As far as the author knows, the results for $n > 1$ relative to ours were only limited to $n = 3$ before 1980 (cf. Singare and Pachpatte [14–15]). In 1985, Yeh [20–21] generalized these results to n independent variables using many techniques. However, as applications of our results we can easily get some new discrete inequalities of the Gronwall–Bellman–Bihari type in several variables which include and generalize the recent known results of Singare and Pachpatte [13–14] and Yeh [20–21].

We first give some notations. Let N be the set of nonnegative integers. For $i = (i_1, \dots, i_n)$, $j = (j_1, \dots, j_n)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{0} = (0, \dots, 0) \in N^n$, we define

$$\sum_{j=0}^{i-1} u(j) = \sum_{j_1=0}^{i_1-1} \cdots \sum_{j_n=0}^{i_n-1} u(j_1, \dots, j_n).$$

We let $\Delta_{i=0}^1 \delta(i) = 0$. Let $I = (i_1, \dots, i_n) \in N$ be fixed with $0 \leq i$. We will use notations $\{0, I\} := \{i \in N^n: 0 \leq i \leq I\}$, $[j, j+1) := \{x \in R^n: j \leq x < j+1\}$, $(j, j+1] := \{x \in R^n: j < x \leq j+1\}$ and so on. We also denote by $\mathcal{F}_2$ the family of all function $\alpha: \{0, I\} \rightarrow \{0, I\}$ such that $\alpha(i) \leq i$ for all $i \in \{0, I\}$.

We first use Theorem 3.1 to prove the following theorem which generalizes Theorem 1 of Yeh [20], hence, Singare and Pachpatte [14, Theorem 3] and [15, Theorem 1].

Theorem 4.1. *Let $\xi(i)$, $\mu(i)$, $\zeta(i)$ and $\gamma(i)$ be nonnegative functions defined on $\{0, I\}$ with $\xi(i) \geq 1$ for all $i \in \{0, I\}$. Let $\eta(i)$ be a positive and nondecreasing function defined on $\{0, I\}$. Let $\alpha(i) \in \mathcal{F}_2$ and $H \in \mathcal{H}$ with corresponding multiplier function Φ . Then inequality*

$$\xi(i) \leq \eta(i) + \zeta(i) \sum_{j=0}^{i-1} \mu(j) H \left(\xi(\alpha(j)) + \zeta(\alpha(j)) \sum_{l=0}^{\alpha(j)} \gamma(l) H(\xi(\alpha(j))) \right) \quad (39)$$

for all $i \in \{0, I\}$, implies

$$\xi(i) \leq \eta(i) \zeta(i) G^{-1} \left(G(1) + \sum_{j=0}^{i-1} [\mu(j) + \gamma(j)] \eta(j) \Phi(\zeta(\alpha(j)) \eta(\alpha(j))) \right) \quad (40)$$

for all $i \in \{0, i^*\}$, where G and G^{-1} are as defined in Lemma 2.2 and $i^* \in \{0, I\}$ is such that

$$G(1) + \sum_{j=0}^{i-1} [\mu(j) \eta(j)] \Phi(\zeta(\alpha(j)) \eta(\alpha(j))) \in \text{Dom}(G^{-1}), \quad i \in \{0, i^*\}.$$

Proof. Define $\beta(i) \in \mathcal{F}_2$ as follows:

$$\beta(i) := \begin{cases} 0, & i \in \{0, 1\} \\ \alpha(i-1) + 1, & i \in \{1, I\}. \end{cases}$$

Thus, inequality (39) is equivalent to

$$\begin{aligned} \xi(i) &\leq \eta(i) + \zeta(i) \sum_{j=0}^{i-1} \mu(j) H(\xi(\beta(j+1)) - 1) \\ &\quad + \zeta(\beta(j+1) - 1) \sum_{l=0}^{\beta(j+1)-1} \gamma(l) H(\xi(\beta(l+1)) - 1), \quad i \in \{0, I\}. \end{aligned} \quad (41)$$

We now define $A(x)$, $u(x)$, $q(x)$, $f(x)$, $g(x)$ and $\sigma(x)$ for all $0 \leq x \leq I$ as follows:

$$A(x) := (A_1(x_1), \dots, A_n(x_n))$$

$$A_1(x_1) := \sum_{k=1}^{I_1} \chi_{[k, \infty)}(x_1), \quad l = 1, 2, \dots, n$$

$$u(x) := \sum_{j=0}^I \xi(j) \chi_{[j, j+1)}(x), \quad f(x) := \sum_{j=0}^I \mu(j) \chi_{(j, j+1)}(x),$$

$$g(x) := \sum_{j=0}^I \gamma(j) \chi_{[j, j+1)}(x), \quad n(x) := \sum_{j=0}^I \eta(j) \chi_{[j, j+1)}(x),$$

$$q(x) := \sum_{j=0}^I \zeta(j) \chi_{[j, j+1)}(x), \quad \sigma(x) := \sum_{j=0}^I \beta(j) \chi_{(j, j+1)}(x),$$

$$u(x) \leq n(x) + q(x) \left\{ \int_0^x f(s) H \left[u(\sigma(s) -) + q(\sigma(s) -) \right. \right. \\ \left. \left. \times \int_0^{\sigma(s)-} g(\tau) H(u(\sigma(\tau) -)) dA(\tau) \right] dA(s) \right\}$$

which, in view of Theorem 3.1, implies

$$\xi(i) \leq \eta(i) \zeta(i) G^{-1} \left(G(1) + \sum_{j=0}^{i-1} [(\mu(j) + \gamma(j))/\eta(j)] \right. \\ \left. \times \Phi(\zeta(\beta(j+1) - 1) \eta(\beta(j+1) - 1)) \right)$$

for all $i \in \{0, i^*\}$. This is desired result (40) which completes the proof.

Similarly, we can apply Theorems 3.2–3.5 to prove the following theorems.

Theorem 4.2. Let $\xi(i)$, $\eta(i)$, $\zeta(i)$, $\mu(i)$ and $\gamma(i)$ be nonnegative functions defined on $\{0, I\}$. Let $\alpha(i) \in \mathcal{F}_2$ and $K \in \mathcal{K}$ with corresponding multiplier function Φ and addition function φ . Then, inequality

$$\xi(i) \leq \eta(i) + \zeta(i) \left\{ \sum_{j=0}^{i-1} \mu(j) K \left(\xi(\alpha(j)) + \zeta(\alpha(j)) \sum_{l=0}^{\alpha(j)} \gamma(l) K(\xi(\alpha(l))) \right) \right\} \quad (42)$$

for all $i \in \{0, I\}$, implies

$$\xi(i) \leq \eta(i) + \zeta(i) F^{-1} \left\{ F \left(\sum_{j=0}^{i-1} (\mu(j) + \gamma(j)) \varphi(\eta(\alpha(j))) \right) \right. \\ \left. + \sum_{j=0}^{i-1} (\mu(j) + \gamma(j)) \Phi(\zeta(\alpha(j))) \right\}, \quad i \in \{0, i^*\} \quad (43)$$

where F and F^{-1} are as defined in Theorem 3.2, and $i^* \in \{0, I\}$ is chosen so that

$$F \left(\sum_{j=0}^{i-1} (\mu(j) + \gamma(j)) \varphi(\eta(\alpha(j))) \right) + \sum_{j=0}^{i-1} (\mu(j) + \gamma(j)) \Phi(\zeta(\alpha(j))) \in \text{Dom}(F^{-1})$$

for all $i \in \{0, i^*\}$.

Remark 4.1. If $\sigma(i) = i$, K is subadditive and submultiplicative, Theorem 4.2 reduces to Theorem 2 of Yeh [20] which in turn is an improvement of Theorem 2 of Singare and Pachpatte [14].

Theorem 4.3. Let $\xi(i)$, $\eta(i)$, $\mu(i)$, $\alpha(i)$ be defined as in Theorem 4.2 and $\xi_0 > 0$. Let P , P^{-1} , Q , U , U^{-1} be defined as in Theorem 3.3. If the inequality

$$\xi(i) \leq \eta(i) + P^{-1} \left(Q \left\{ \xi_0 + \sum_{j=0}^{i-1} \mu(j) P(\xi(\alpha(j))) \right\} \right)$$

holds for all $i \in \{0, I\}$, we then have

$$\xi(i) \leq \eta(i) P^{-1} \left(1 + Q \left\{ U^{-1} \left[U(\xi_0) + \sum_{j=0}^{i-1} \mu(j) P(\eta(\alpha(j))) / P(\eta(j)) \right] \right\} \right)$$

for all $i \in \{0, i^*\}$, where $i^* \in \{0, I\}$ is such that

$$U(\xi_0) + \sum_{j=0}^{i-1} \mu(j) P(\eta(\alpha(j))) / P(\eta(j)) \in \text{Dom}(U^{-1}), \quad i \in \{0, i^*\}$$

Remark 4.2. If $\alpha(i) = i$ and $\xi_0 = 0$, Theorem 4.3 reduces to Theorem 3 of Yeh [20].

Theorem 4.4. Let $\xi(i)$, $\zeta(i)$, $\mu(i)$, $\gamma(i)$, $\alpha(i)$ be defined as in Theorem 4.2 with $\zeta(i) \geq 1$ for all $i \in \{0, I\}$. Let $\xi_0 > 0$ and H, G, G^{-1} be defined as in Theorem 3.4. Then inequality

$$\xi(i) \leq \xi_0 + \zeta(i) \sum_{j=0}^{i-1} \mu(j) H(\xi(\alpha(j))) + \sum_{j=0}^{i-1} \gamma(j) \sum_{l=0}^{j-1} \mu(l) H(\xi(\alpha(l)))$$

for all $i \in \{0, I\}$, implies

$$\xi(i) \leq \zeta(i) E(i) G^{-1} \left(G(\xi_0) + \sum_{j=0}^{i-1} \mu(j) \Phi(\zeta(\alpha(j))) E(\alpha(j)) \right)$$

for all $i \in \{0, i^*\}$, where

$$E(i) = \exp \left\{ \sum_{j=0}^{i-1} \gamma(j) / \zeta(j) \right\}$$

$i^* \in \{0, I\}$ is such that

$$G(\xi_0) + \sum_{j=0}^{i-1} \mu(j) \Phi(\zeta(\alpha(j))) E(\alpha(j)) \in \text{Dom}(G^{-1}), \quad i \in \{0, i^*\}.$$

Theorem 4.5. Let $\xi(i)$, $\eta(i)$, $\zeta(i)$, $\mu(i)$, $\gamma(i)$ be defined as in Theorem 4.2 and H be defined as in Theorem 3.5. Suppose

$$\xi(i) \leq \eta(i) + \zeta(i) \left(\sum_{j=0}^{i-1} \mu(j) H(\xi(j)) + \sum_{j=0}^{i-1} \gamma(j) \sum_{l=0}^{j-1} \mu(l) H(\xi(l)) \right)$$

holds for all $i \in \{0, I\}$. Then

$$\xi(i) \leq \eta(i) + \zeta(i) \exp \left(\sum_{j=0}^{i-1} [k\eta(j)\zeta(j) + \gamma(j)] \right) \sum_{j=0}^{i-1} \mu(j) H(\eta(j))$$

for all $i \in \{0, I\}$.

Since the proofs of Theorems 4.2–4.5 are all similar to the proof of Theorem 4.1, we omit them and our paper is complete.

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Integrability and L^1 -convergence of sine series with generalized quasi-convex coefficients

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Abstract. In this paper we obtain a necessary and sufficient condition for the sine series with generalized quasi-convex coefficients to be a Fourier series. Also we study L^1 -convergence of this series under the said condition on the coefficients.

Keywords. Quasi-convex sequence; generalized quasi-convex sequence; L^1 -convergence.

1. Introduction

Let $\langle a_k \rangle$, $k = 0, 1, 2, \dots$, be a quasi-convex sequence of real numbers tending to zero, that is, $a_k \rightarrow 0$ as $k \rightarrow \infty$ and

$$\sum_{k=1}^{\infty} k |\Delta^2 a_{k-1}| < \infty, \quad (1)$$

where

$$\Delta^2 a_{k-1} = \Delta a_{k-1} - \Delta a_k = a_{k-1} - 2a_k + a_{k+1}.$$

Then the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \quad (2)$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx \quad (3)$$

are uniformly convergent on every segment $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$, and are therefore convergent for $0 < x < 2\pi$.

It is known (cf. [1, Vol II, p. 202]) that if the null sequence satisfies (1), then (2) is a Fourier series and

$$\int_0^\pi \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leq \frac{\pi}{2} \sum_{k=1}^{\infty} k |\Delta^2 a_{k-1}|.$$

Moore [3] generalized the notion of quasi-convex sequence in the following way: A null sequence $\langle a_k \rangle$ is said to be generalized quasi-convex if

$$\sum_{k=1}^{\infty} k^\alpha |\Delta^{\alpha+1} a_k| < \infty, \text{ for } \alpha > 0. \quad (4)$$

concerning the integrability of cosine series (2):

Theorem A. *If $\langle a_k \rangle$ is a null sequence satisfying (4), then the series (2) converges in the interval $0 < x \leq \pi$ and represents there an L -integrable function whose Fourier cosine development is given by (2).*

Teljakovskii [5] considered the problem as to when the series (3) will be Fourier series, that is, when will the function to which the series (3) converges be integrable. Assuming $a_0 = 0$, he established the following theorem:

Theorem B. *If $\langle a_k \rangle$ is a quasi-convex null sequence, then (3) is a Fourier series if and only if*

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty. \quad (5)$$

Moreover, if the series in (5) converges, then

$$\left| \int_0^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx - \sum_{k=1}^{\infty} \frac{|a_k|}{k} \right| \leq C \sum_{k=1}^{\infty} k |\Delta^2 a_{k-1}|,$$

where C is an absolute constant.

Concerning the L^1 -convergence of cosine series, Singh and Sharma [4] established the following theorem:

Theorem C. *Let k be a real number such that $k > 0$. If $\langle a_k \rangle$ be a null sequence satisfying (4), then for L^1 -convergence of the series (2), it is necessary and sufficient that $\lim_{k \rightarrow \infty} a_k \log k = 0$.*

The object of this paper is to generalize Theorem B for the sine series with generalized quasi-convex coefficients. Also we shall study L^1 -convergence of this series under the said condition on the coefficients.

In what follows, we shall use the following notation:

Given a sequence $S_0, S_1, S_2, \dots$, we define for every $k = 0, 1, 2, \dots$, the sequence $S_0^k, S_1^k, S_2^k, \dots$, by the conditions

$$S_n^0 = S_n, S_n^k = S_0^{k-1} + S_1^{k-1} + \dots + S_n^{k-1} \quad \begin{pmatrix} k = 1, 2, \dots \\ n = 0, 1, 2, \dots \end{pmatrix}.$$

Similarly for $k = 0, 1, 2, \dots$, we define [6] the sequence of numbers $A_0^k, A_1^k, A_2^k, \dots$ by the conditions

$$A_n^0 = 1, A_n^k = A_1^{k-1} + A_1^{k-1} + \dots + A_n^{k-1} \quad \begin{pmatrix} k = 1, 2, \dots \\ n = 0, 1, 2, \dots \end{pmatrix}.$$

The A_n 's are called binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty} A_k^{\alpha} x^k = (1-x)^{-\alpha-1},$$

whereas S_n 's are given by

$$\sum_{k=0}^{\infty} S_k^{\alpha} x^k = (1-x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^k.$$

We shall use the following relations:

$$A_n^{\alpha} = \sum_{v=0}^n A_v^{\alpha-1}, \quad A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1},$$

$$A_n^{\alpha} = \binom{n+\alpha}{n} \simeq \frac{n^{\alpha}}{\sqrt{(\alpha+1)}} \quad (\alpha \neq 1, -2, \dots).$$

The Cesaro means T_k^{α} of order α of Σa_n will be defined by $T_k^{\alpha} = S_k^{\alpha}/A_k^{\alpha}$.

2. Lemma

The following lemma will be used for the proofs of our results.

Lemma 1. [2] *If for $\alpha \geq 0$, $p \geq 0$,*

$$(i) \quad \varepsilon_n = O(n^{-p}); \quad (ii) \quad \sum_{n=0}^{\infty} A_n^{\alpha+p} |\Delta^{\alpha+1} \varepsilon_n| < \infty,$$

then

$$(iii) \quad \sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \varepsilon_n| < \infty \text{ for } -1 \leq \lambda \leq \alpha, \text{ and}$$

(iv) $A_n^{\lambda+p} \Delta^{\lambda} \varepsilon_n$ is of bounded variation for $0 \leq \lambda \leq \alpha$ and tends to zero as $n \rightarrow \infty$.

3. Results

We prove the following results:

Theorem 1. *Let $\langle a_k \rangle$, $k = 0, 1, 2, \dots$, $a_0 = 0$ be a generalized quasi-convex null sequence. Then (3) is a Fourier series if and only if (5) is satisfied.*

Moreover, if (5) holds, then

$$\left| \int_0^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx - \sum_{k=1}^{\infty} \frac{|a_k|}{k} \right| \leq C \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1}|,$$

where C is an absolute constant not necessarily the same at each occurrence.

The assertions of Theorem 1 are immediate consequences of the following theorem:

Theorem 2. *Let $\langle a_k \rangle$, $k = 0, 1, 2, \dots$, $a_0 = 0$, be a generalized quasi-convex null sequence. Then for all $s = 1, 2, \dots$, we have*

$$\left| \int_{1/(s+1)}^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx - \sum_{k=1}^s \frac{|a_k|}{k} \right| \leq C \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1}|.$$

Proof of Theorem 2. Let $0 < x \leq \pi$ and let

$$\bar{D}_0(x) = -\frac{1}{2} \cot \frac{x}{2}$$

$$\bar{S}_n(x) = \bar{D}_0(x) + \sin x + \sin 2x + \cdots + \sin nx$$

$$\bar{S}_n^1(x) = \bar{S}_0(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \cdots + \bar{S}_n^1(x)$$

$$\bar{S}_n^2(x) = \bar{S}_0^1(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \cdots + \bar{S}_n^1(x)$$

$$\bar{S}_n^k(x) = \bar{S}_0^{k-1}(x) + \bar{S}_1^{k-1}(x) + \cdots + \bar{S}_n^{k-1}(x).$$

$$\bar{T}_n^k = \frac{\bar{S}_n^k}{A_n^k} = -\frac{1}{A_n^k} \sum_{v=0}^n A_{n-v} \frac{\cos(v+1/2)x}{2 \sin x/2} \quad (1/n \leq x < \pi)$$

Now

$$\begin{aligned} \sum_{k=0}^n a_k \sin kx &= \sum_{k=0}^{n-1} \Delta a_k \left(\sum_{v=0}^k \sin vx \right) + a_n \sum_{v=0}^n \sin vx \\ &= \sum_{k=0}^{n-1} \Delta a_k (\bar{S}_k(x) - \bar{S}_0(x)) + a_n (\bar{S}_n(x) - \bar{S}_0(x)) \\ &= \sum_{k=0}^{n-1} \Delta a_k \bar{S}_k(x) - \sum_{k=0}^{n-1} \Delta a_k \bar{S}_0(x) + a_n \bar{S}_n(x) - a_n \bar{S}_0(x) \\ &= \sum_{k=0}^{n-1} \Delta a_k \bar{S}_k(x) + a_n \bar{S}_n(x) - a_0 \bar{S}_0(x) \\ &= \sum_{k=0}^{n-1} \Delta a_k \bar{S}_k(x) + a_n \bar{S}_n(x) \\ &= \sum_{k=0}^{n-2} \Delta^2 a_k \bar{S}_k^1(x) + \Delta a_{n-1} \bar{S}_{n-1}^1(x) + a_n \bar{S}_n(x). \end{aligned}$$

If we use Abel's transformation $(\alpha + 1)$ times, we have similarly,

$$\begin{aligned} \sum_{k=0}^n a_n \sin kx &= \sum_{k=0}^{n-\alpha-1} \Delta^{\alpha+1} a_k \bar{S}_k^\alpha(x) + \Delta^\alpha a_{n-\alpha} \bar{S}_{n-\alpha}^\alpha(x) + \cdots \\ &\quad + \Delta^2 a_{n-2} \bar{S}_{n-2}^2(x) + \Delta a_{n-1} \bar{S}_{n-1}^1(x) + a_n \bar{S}_n(x) \\ &= \sum_{k=0}^{n-\alpha-1} A_k^\alpha \Delta^{\alpha+1} a_k \bar{T}_k^\alpha + A_{n-\alpha}^\alpha \Delta^\alpha a_{n-\alpha} \bar{T}_{n-\alpha}^\alpha + \cdots \\ &\quad + A_{n-2}^2 \Delta^2 a_{n-2} \bar{T}_{n-2}^2 + A_{n-1}^1 \Delta a_{n-1} \bar{T}_{n-1}^1 + a_n \bar{S}_n(x). \end{aligned}$$

Since $\bar{S}_n(x)$ and $\bar{T}_n^\alpha(x)$ are uniformly bounded on every segment $[\varepsilon, \pi]$, $\varepsilon > 0$, it

follows, by Lemma 1, that

$$\begin{aligned}\sum_{k=1}^{\infty} a_n \sin kx &= \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} A_{k-1}^{\alpha} \bar{T}_{k-1}^{\alpha} \\ &= \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \bar{S}_{k-1}^{\alpha}(x).\end{aligned}\quad (7)$$

We introduce the auxiliary function

$$\gamma(U) = \begin{cases} U & \text{for } 0 \leq U \leq 1 \\ 1 & \text{for } 1 \leq U < \infty \end{cases}$$

We have, by our notation,

$$\bar{S}_{k-1}^{\alpha}(x) = - \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-1} \frac{\cos(v+1/2)x}{2 \sin x/2}.$$

Applying Abel's transformation, we obtain

$$\begin{aligned}\bar{S}_{k-1}^{\alpha}(x) &= - \sum_{v=0}^{k-2} \Delta A_{k-v-1}^{\alpha-1} \frac{\sin(v+1)x}{4 \sin^2 x/2} - A_0^{\alpha-1} \frac{\sin kx}{4 \sin^2 x/2} \\ &= - \sum_{v=0}^{k-2} A_{k-v-1}^{\alpha-2} \frac{\sin(v+1)x}{4 \sin^2 x/2} - A_0^{\alpha-2} \frac{\sin kx}{4 \sin^2 x/2} \\ &= \sum_{v=0}^{k-2} A_{k-v-1}^{\alpha-2} \bar{S}_v^1(x) + A_0^{\alpha-2} \bar{S}_{k-1}^1(x) \\ &= \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \bar{S}_v^1(x).\end{aligned}\quad (8)$$

Thus, from (7) and (8), it follows that

$$\begin{aligned}\sum_{k=1}^{\infty} a_k \sin kx &= \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \bar{S}_v^1(x) \\ &= - \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \frac{\gamma((k-1)x)}{x^2} \\ &\quad + \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \left[\sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right] \\ &= - \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} A_{k-1}^{\alpha-1} \frac{\gamma((k-1)x)}{x^2} \\ &\quad + \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \left[\sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right].\end{aligned}\quad (9)$$

But

$$\begin{aligned}&\int_0^{\pi} \left| \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \left[\sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right] \right| dx \\ &\leq \sum_{k=1}^{\infty} |\Delta^{\alpha+1} a_{k-1}| \int_0^{\pi} \left| \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right| dx.\end{aligned}\quad (10)$$

Now, following Teljakovski [5], we have

$$\begin{aligned} \int_0^\pi \left| \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right| dx \\ = \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \int_0^\pi \left| \bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right| dx \\ < Ck \sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \\ = Ck^\alpha. \end{aligned}$$

The relation (10) then yields

$$\begin{aligned} \int_0^\pi \left| \sum_{k=1}^\infty \Delta^{\alpha+1} a_{k-1} \left[\sum_{v=0}^{k-1} A_{k-v-1}^{\alpha-2} \left(\bar{S}_v^1(x) + \frac{\gamma((k-1)x)}{x^2} \right) \right] \right| dx \\ \leq \sum_{k=1}^\infty k^\alpha |\Delta^{\alpha+1} a_{k-1}|. \end{aligned} \quad (11)$$

Thus, by (9) and (11), we obtain

$$\begin{aligned} \left| \int_{1/(s+1)}^\pi \left| \sum_{k=1}^\infty a_k \sin kx \right| dx - \int_{1/(s+1)}^\pi \left| \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \right| dx \right| \\ \leq C \sum_{k=0}^\infty k^\alpha |\Delta^{\alpha+1} a_{k-1}| \end{aligned} \quad (12)$$

Moreover,

$$\begin{aligned} \int_{1/(s+1)}^\pi \left| \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \right| dx \\ = \int_1^\pi \left| \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \right| dx \\ + \sum_{p=1}^s \int_{1/(p+1)}^{1/p} \left| \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \right| dx. \end{aligned}$$

But

$$\int_1^\pi \left| \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \right| dx \leq \sum_{k=1}^\infty A_{k-1}^{\alpha-1} |\Delta^{\alpha+1} a_{k-1}|. \quad (13)$$

Now, let $1/p + 1 \leq x \leq 1/p$. Then

$$\begin{aligned} \sum_{k=1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} &= \sum_{k=1}^p A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \\ &\quad + \sum_{k=p+1}^\infty A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^p A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{(k-1)}{x} \\
&\quad + \sum_{k=p+1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{1}{x^2} \\
&< \frac{1}{x} \sum_{k=1}^p A_{k-2}^{\alpha} \Delta^{\alpha+1} a_{k-1} \\
&\quad + \frac{1}{x^2} \sum_{k=p+1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1}.
\end{aligned}$$

But, α times application of Abel's transformation yield

$$\begin{aligned}
\sum_{k=1}^p A_{k-2}^{\alpha} \Delta^{\alpha+1} a_{k-1} &= \sum_{k=1}^p \Delta a_{k-1} - \sum_{r=1}^{\alpha} \Delta^r a_p A_{p-2}^{r-1} \\
&= -a_p - \sum_{r=1}^{\alpha} \Delta^r a_p A_{p-2}^{r-1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=p+1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} &= \sum_{k=p+1}^{\infty} \Delta^2 a_{k-1} + \Delta^{\alpha} a_p A_{p-1}^{\alpha-1} + \Delta^{\alpha-1} a_p A_{p-1}^{\alpha-2} \\
&\quad + \cdots + \Delta^2 a_p A_{p-1}^1 \\
&= \Delta a_p + \Delta^2 a_p A_{p-1}^1 + \Delta^3 a_p A_{p-1}^2 + \cdots + \Delta^{\alpha} a_p A_{p-1}^{\alpha-1} \\
&= \sum_{r=1}^{\alpha} \Delta^r a_p A_{p-1}^{r-1}.
\end{aligned}$$

Therefore

$$\sum_{k=1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} < \frac{-a_p}{x} + \sum_{r=1}^{\alpha} \Delta^r a_p \left(\frac{A_{p-1}^{r-1}}{x^2} - \frac{A_{p-2}^{r-1}}{x} \right).$$

Hence

$$\begin{aligned}
&\left| \int_{1/(1+p)}^{1/p} \sum_{k=1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} dx - \int_{1/(p+1)}^{1/p} \frac{|a_p|}{x} dx \right| \\
&\leq \sum_{r=1}^{\alpha} |\Delta^r a_p| \int_{1/(1+p)}^{1/p} \left| \frac{A_{p-1}^{r-1}}{x^2} - \frac{A_{p-2}^{r-1}}{x} \right| dx \\
&= \sum_{r=1}^{\alpha} |\Delta^r a_p| A_{p-1}^{r-1} \int_{1/(p+1)}^{1/p} \left| \frac{1}{x^2} - \frac{(p-1)}{rx} \right| dx \\
&\leq \sum_{r=1}^{\alpha} |\Delta^r a_p| A_{p-1}^{r-1}.
\end{aligned}$$

Now, since

we have

$$\begin{aligned} & \int_{1/(p+1)}^{1/p} \sum_{k=1}^{\infty} A_{k-1}^{\alpha-1} \Delta^{\alpha+1} a_{k-1} \frac{\gamma((k-1)x)}{x^2} \left| dx - \frac{|a_p|}{p} \right| \\ & \leq \frac{|a_p|}{p^2} + \sum_{r=1}^{\alpha} A_{p-1}^{r-1} \Delta^r a_p \leq \frac{|a_p|}{p^2} + C \sum_{r=1}^{\alpha} p^{r-1} |\Delta^r a_p|. \end{aligned} \quad (14)$$

Since $a_k \rightarrow 0$ as $k \rightarrow \infty$, we have

$$\max |a_k| \leq \sum_{k=0}^{\infty} |\Delta a_k| \leq \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1}|,$$

which yields

$$\sum_{p=1}^{\infty} \frac{|a_p|}{p^2} \leq C \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1}|,$$

the last step being the consequence of the use of Abel's transformation a times.

Moreover,

$$\sum_{r=1}^{\alpha} p^{r-1} |\Delta^r a_p| = |\Delta a_p| + p |\Delta^2 a_p| + p^2 |\Delta^3 a_p| + \cdots + p^{\alpha-1} |\Delta^{\alpha} a_p|$$

and application of Abel's transformation for $\alpha, \alpha-1, \dots, 2, 1$ time, gives

$$\begin{aligned} \sum_{p=1}^s \left(\sum_{r=1}^{\alpha} p^{r-1} |\Delta^r a_p| \right) &= \sum_{p=1}^s |\Delta a_p| + \sum_{p=1}^s p |\Delta^2 a_p| + \cdots + \sum_{p=1}^s p^{\alpha-1} |\Delta^{\alpha} a_p| \\ &\leq \sum_{p=1}^{\infty} p^{\alpha} |\Delta^{\alpha+1} a_p|. \end{aligned}$$

Thus

$$\sum_{p=1}^s \left(\frac{|a_p|}{p^2} + C \sum_{r=1}^{\alpha} p^{r-1} |\Delta^r a_p| \right) \leq C \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1}|. \quad (15)$$

The result now follows from the relations (12), (13), (14) and (15).

Proof of Theorem 3. Suppose that the conditions (5) and (6) are satisfied. By Theorem 1, the condition (5) together with the fact that $a_k \rightarrow 0$ $k \rightarrow \infty$, implies that the sine series is a Fourier series of an $f \in L$. On the other hand, under the given conditions, the series (2) is the Fourier series of an $\tilde{f} \in L$, by Theorem A and it converges in L by Theorem C. Hence the sine series also converges in L (cf. [1], Vol. II, p. 147).

Conversely, suppose that the sine series converges in L . Therefore it is a Fourier series (cf. [1], Vol. I, Chap. I, §12). Hence again by Theorem 1, condition (5) is satisfied. On the other hand, by Theorem A, the series (2) is a Fourier series of an $\tilde{f} \in L$ whence follows the convergence in L of (2). Hence by Theorem C, condition (6) is satisfied.

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Some results involving Bessel function and Fox's H-function of two variables

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Abstract. In this paper, we have presented two integrals and employed them to establish one Fourier Bessel expansion for Fox's H-function of two variables.

Keywords. Fox's H-function; Bessel function; expansion formulae; Fourier series; special functions.

1. Introduction

The object of this paper is to establish one Fourier Bessel expansion for Fox's H-function of two variables with the help of two integrals presented in this paper.

The subject of expansion formulae and Fourier series for generalized hypergeometric functions of two variables occupies a prominent place in the literature of special functions and two-dimensional boundary value problems.

The results presented in this paper are of a very general character and their special cases are scattered throughout the literature. Only two well-known results earlier given by Bajpai [1] are mentioned as particular cases.

The Fox's H-function of two variables is defined and represented as follows [4, p. 82, (6.1.1)]:

$$H \left[\begin{matrix} \zeta \\ \eta \end{matrix} \right] = H_{p_1, q_1; p_2, q_2; p_3, q_3}^{a, n_1; m_2, n_2, n_3} \left[\begin{matrix} \zeta (a_j; \alpha_j, A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3} \\ \eta (b_j; \beta_j, B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{matrix} \right] \quad (1)$$

2. The integrals

The following integrals are required in the proofs of the Fourier Bessel expansions

$$\int_0^\infty x^{\rho-1} J_\nu(ax) H \left[\begin{matrix} \zeta x^{2h} \\ \eta \end{matrix} \right] dx = \frac{2^{\rho-1}}{a^\rho} H_{p_1, q_1; p_2+2, q_2; p_3, q_3}^{a, n_1; m_2, n_2+1; m_3, n_3} \left[\begin{matrix} \left(\frac{2}{a} \right)^{2h} \zeta (a_j; \alpha_j, A_j)_{1, p_1}; \\ \eta (b_j; \beta_j, B_j)_{1, q_1}; \\ \left(\frac{2-\rho-\nu}{2}, h \right), (c_j, \gamma_j)_{1, p_2}, \left(\frac{2-p+\nu}{2}, h \right); (e_j, E_j)_{1, p_3} \\ (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{matrix} \right], \quad (2)$$

$$\operatorname{Re}(\rho + \nu) + 2h \min_{1 \leq j \leq m_2} [\operatorname{Re} d_j / \delta_j] > 0, \operatorname{Re} \rho + 2h \max_{1 \leq j \leq n_2} \\ \times [\operatorname{Re}(c_j - 1) / \gamma_j] < \frac{1}{2}, a > 0, h > 0;$$

$$\int_0^\infty y^{\sigma-1} J_\mu(ay) H \left[\begin{matrix} \zeta \\ \eta y^{2k} \end{matrix} \right] dy = \frac{2^{\sigma-1}}{a^\sigma} H_{p_1, q_1; p_2, q_2; p_3+2, q_3}^{o, n_1, m_2, n_2+1; m_3, n_3+1} \left[\begin{matrix} \zeta \\ \left(\frac{2}{a}\right)^{2k} \eta \end{matrix} \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1}; \\ (b_j; \beta_j, B_j)_{1, q_1}; \\ (c_j, \gamma_j)_{1, p_2}; \left(\frac{2-\sigma-\mu}{2}, k\right); (e_j, E_j)_{1, p_3}; \left(\frac{2-\sigma+\mu}{2}, k\right) \end{matrix} \right. \right. \\ \left. \times \left. \begin{matrix} (d_j, \delta_j)_{1, q_2}; (\delta_j, F_j)_{1, q_3} \end{matrix} \right] \right], \quad (3) \\ \operatorname{Re}(\sigma + \mu) + 2k \min_{1 \leq j \leq m_3} [\operatorname{Re} f_j / F_j] > 0, \operatorname{Re} \sigma + 2k \max_{1 \leq j \leq n_3} \\ \times [\operatorname{Re}(e_j - 1) / E_j] < \frac{1}{2}, a > 0; k > 0;$$

and the conditions given by [4, p. 83, (6.1.5) and (6.1.6)] are also satisfied.

Proof. To establish (2), expressing the H-function in the integrand as [4, p. 83, (6.1.1)], changing the order of the x-integral and (s, t)-integral, evaluating the inner-integral with the help of [2, p. 22, (7)] and using [4, p. 83, (6.1.1)], the value of the integral (2) is obtained.

On using the same procedure as above, the integral (2) is established.

Note. The integrals (2) and (3) may be viewed as the analogues of the integral [1, p. 18, (2.1)].

3. Fourier Bessel expansion

The Fourier Bessel expansion to be established as

$$x^\rho y^\sigma H \left[\begin{matrix} \zeta x^{2h} \\ \eta y^{2k} \end{matrix} \right] = \sum_{r, t=0}^{\infty} \frac{(2r + \lambda + 1)(2t + u + 1)}{2^{-\rho-\sigma}} H_{p_1, q_1; p_2, q_2; p_3+2, q_3}^{o, n_1; m_2, n_2+1; m_3, n_3+1} \\ \times \left[\begin{matrix} 2^{2h} \zeta \\ 2^{2k} \eta \end{matrix} \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1}; \left(\frac{1-\rho-\lambda-2r}{2}, h\right); (c_j, \gamma_j)_{1, p_2}; \\ (b_j; \beta_j, B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{matrix} \right. \right. \\ \left. \left. \left(\frac{3-\rho+u+2r}{2}, h\right); \left(\frac{1-\sigma-u-2t}{2}, k\right); (e_j, E_j)_{1, p_3}; \right. \right. \\ \left. \left. \left(\frac{3-\sigma+u+2t}{2}, k\right) \right] J_{\lambda+2r+1}(x) J_{u+2t+1}(y), \quad (4)$$

$$\operatorname{Re}(\rho + \lambda) + 2h \min_{1 \leq j \leq m_2} [\operatorname{Re} d_j / \delta_j] > -1, \operatorname{Re} \rho + 2h \max_{1 \leq j \leq n_2}$$

$$[\operatorname{Re}(c_j - 1) / \gamma_j] < \frac{1}{2}, \operatorname{Re}(\sigma + u) + 2k \min_{1 \leq j \leq m_3} [\operatorname{Re} f_j / F_j] > -1,$$

$$\operatorname{Re} \sigma + 2k \max_{1 \leq j \leq n_3} [\operatorname{Re}(e_j - 1) / E_j] < \frac{1}{2}, h > 0, k > 0.$$

and the conditions given by [4, p. 83, (6.1.5) and (6.1.6)] are also satisfied.

Proof. To establish (4) let

$$f(x, y) = x^\rho y^\sigma H \left[\frac{\zeta x^{2h}}{\eta y^{2k}} \right] = \sum_{r,t=0}^{\infty} A_{r,t} J_{\lambda+2r+1}(x) J_{u+2t+1}(y). \quad (5)$$

Equation (5) is valid, since $f(x, y)$ is continuous and of bounded variation in the region $0 < x < \infty$, $0 < y < \infty$.

Multiplying both sides of (5) by $y^{-1} J_{u+2n+1}(y)$ and integrating with respect to y from 0 to ∞ and using (3) and the orthogonality property of Bessel functions [3, p. 291, (6)], viz.

$$\begin{aligned} \int_0^{\infty} x^{-1} J_{v+2n+1}(x) J_{v+2m+1}(x) dx &= 0, \text{ if } m \neq n \\ &= (4n+2v+2)^{-1}, \text{ if } m = n, \operatorname{Re} v > -1; \end{aligned} \quad (6)$$

we have

$$\begin{aligned} &2^\rho x^\rho H_{p_1, q_1; m_2, n_2; m_3, n_3+1}^{0, n_1; p_2, q_2; p_3+2, q_3} \left[\frac{\zeta x^{2h}}{2^{2k} \eta} \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1}; & (c_j; \gamma_j)_{1, p_2}; \\ (b_j; \beta_j, B_j)_{1, q_1}; & (d_j; \delta_j)_{1, q_2}; \end{matrix} \right. \right. \\ &\quad \left. \left(\frac{1-\sigma-u-2n}{2}, k \right), (e_j, E_j)_{1, p_3}, \left(\frac{3-\sigma+u+2n}{2}, k \right) \right] \\ &\quad \left. (f_j, F_j)_{1, q_3} \right] \\ &= \sum_{r=0}^{\infty} A_{r,n} \frac{1}{(2n+u+1)} J_{\lambda+2r+1}(x). \end{aligned} \quad (7)$$

Multiplying both sides of (7) by $x^{-1} J_{\lambda+2m+1}(x)$ and integrating with respect to x from 0 to ∞ and using (2) and (6) we get

$$\begin{aligned} A_{m,n} &= \frac{(2m+\lambda+1)(2n+u+1)}{2^{-\rho-\sigma}} \\ &\quad \times H_{p_1, q_1; m_2, n_2+1; m_3, n_3+1}^{0, n_1; p_2, q_2; p_3+2, q_3} \left[\frac{2^{2h} \zeta}{2^{2k} \eta} \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1}; \\ (b_j; \beta_j, B_j)_{1, q_1}; \end{matrix} \right. \right. \\ &\quad \left. \left(\frac{1-\rho-\lambda-2m}{2}, h \right), (c_j, \gamma_j)_{1, q_2}, \left(\frac{3-\rho+\lambda+2m}{2}, h \right); \right. \\ &\quad \left. (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \right. \\ &\quad \left. \left(\frac{1-\sigma-u-2n}{2}, k \right), (e_j, E_j)_{1, p_3}, \left(\frac{3-\sigma+u+2n}{2}, k \right) \right]. \end{aligned} \quad (8)$$

4. Particular cases

On specializing the parameters of the H-function of two variables in the results (2) and (4), we obtain two known results given by Bajpai [1, pp. 18–19, (2.1) and (3.1)].

Note. The results analogous to our main results (4) involving Fox's H-function of several variables can be derived easily on following the technique given in this paper.

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Köthe spaces and topological algebra with bases

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Abstract. Nuclear Köthe sequence space $\lambda(P)$, its crossdual $\lambda(P)^*$ and their non-nuclear variants are examined as topological algebras. Modelling on them, a general theory of nuclear topological algebras with orthogonal basis is developed. As a by-product, abstract characterizations of sequence algebras ℓ^∞ and c_0 are obtained. In a topological algebra set-up, an abstract Grothendieck-Pietsch nuclearity criterion is developed.

Keywords. Köthe sequence spaces; G_1 - and G_∞ -conditions; cross-dual of a sequence space; nuclear locally convex space; Schauder basis; equicontinuous and absolute basis; unconditional convergence; locally convex algebra; orthogonal basis; Grothendieck-Pietsch nuclearity criteria.

1. Introduction

Let ω denote the vector space of all sequences of complex numbers, A subset $P \subset \omega$ is a *power set* [15, Ch. 6] if (i) for each $a = (a_n) \in P$, $a_n \geq 0$, (ii) for each $a = (a_n)$, each $b = (b_n)$ in P , there exists a $c = (c_n)$ in P satisfying $a_n \leq c_n$, $b_n \leq c_n$ for all n , (iii) for each $n \in \mathbb{N}$, there exists $a \in P$ such that $a_n > 0$. Further, P satisfies G_1 -property (respectively G_∞ -property) [12] if (iv) for each $a \in P$, $a_{n+1} \leq a_n$ (respectively $a_n \leq a_{n+1}$) for all n ; (v) for each $a \in P$, there exists a $d \in P$ satisfying $a_n \leq d_n^2$ (respectively $a_n^2 \leq d_n$) for all n . Given a power set P , Köthe sequence space $\lambda(P) = \{b = (b_n) \in \omega \mid p_a(b) = \sum_{n \geq 1} |b_n| a_n < \infty \text{ for all } a \in P\}$ is a complete locally convex space with the Köthe topology τ_P , defined by seminorms $\Gamma_P = \{p_a : a \in P\}$ and having Schauder basis $e_n = (\delta_{nm})$, δ_{nm} being the Kronecker delta [12], [13]. For a G_1 (respectively G_∞) power set P , $\lambda(P)$ is denoted by $\lambda_1(P)$ (respectively $\lambda_\infty(P)$). Note that $\lambda(P)$ is a very general construction that includes very many standard sequence spaces as particular cases [16]. The space $\lambda(P)$ occupies a central position in the theory of locally convex space with basis ([14], [12], [15]); in fact, a complete locally convex space with an absolute basis has to be $\lambda(P)$ [15, Th. 10.1.4].

Now consider ω as a unital $*$ -algebra with pointwise multiplication and pointwise complex conjugation as the involution. It happens that each of $\lambda_1(P)$ and $\lambda_\infty(P)$ (assumed nuclear) is a $*$ -subalgebra of ω which, with topology τ_P , is a locally convex algebra. The present paper is aimed at understanding these topological algebras and examining their relevance in (and impact on) the general axiomatics of functional analysis. It so turns out that $\lambda(P)$ also occupies a central position in the theory of topological algebras with orthogonal basis initiated by Husain and his colleagues [7]–[10].

is shown in Theorem 2.4 that a sequentially complete topological algebra (as a locally convex space) topological algebra A with jointly continuous multiplication and having an equicontinuous orthogonal basis is homeomorphically isomorphic to a (semi) G_1 -Köthe algebra $\lambda(P)$. This provides a topological algebra analogue of [15, Th. 10.1.4] which implies that the space $\lambda(P)$ comprise all complete nuclear spaces with equicontinuous basis. This also generalizes a characterization, due to Husain and Watson [8], of the algebra ω (pointwise convergence) as the only complete unital locally m -convex algebra with orthogonal basis. The relation between nuclearity of A and existence of identity in A is examined in Corollary 2.6, as well as the relevance of identity in above quoted result of Husain and Watson is also discussed. Even in the absence of validity of open mapping theorem, it turns out that an orthogonal basis in a topological algebra is Schauder. In fact, in Proposition 2.1, we prove a much stronger assertion improving [5, Th. 1.1]. This leads to a characterization of isomorphisms of B_0 -algebras with bases. It is shown, in Theorem 2.13, that under fairly general conditions, a sequentially complete topological algebra with an orthogonal basis is a countable direct sum of simple algebras, each homeomorphically isomorphic to the field ϕ of complex numbers. This improves a structure theorem for Banach algebras with basis proved in [10].

Section 3 contains abstract characterizations of Banach algebras ℓ^∞ and c_0 . A unital dual Banach algebra A , with predual A_* , which, for the topology $\sigma(A, A_*)$, (i) is sequentially complete, and (ii) admits an equicontinuous orthogonal basis, is dual algebra isomorphic to ℓ^∞ . Further, such an A is a uniform (Banach) algebra iff this isomorphism is an isometry. On the other hand, a uniform Banach algebra with an orthogonal basis is isometrically isomorphic to c_0 .

In §4, nuclearity of a topological algebra with basis is characterized in terms of ring theoretic structure. Recall the Grothendieck-Pietsch nuclearity criteria [12], [15] viz $(\lambda(P), \tau_p)$ is nuclear iff for each $a \in P$, there exists $ab \in P$ and $au \in \ell^1$ such that $a_n \leq u_n b_n$ for all n iff τ_p is determined by the family of seminorms $\{q_a | a \in P\}$, $q_a(x) = \sup_n |x_n| a_n$. Further, $(\lambda_1(P), \tau_p)$ is nuclear iff $P \subset \ell^1$; and $(\lambda_\infty(P), \tau_p)$ is nuclear iff there is a $u \in P$ such that $(1/u_n) \in \ell^1$. With a view of obtaining an abstract version of this for a topological algebra A with an orthogonal basis (e_n) , we consider, for $1 \leq p \leq \infty$, $K_p(A) = \{x \in A | (e_n^*(xy)) \in \ell^p \text{ for all } y \in A\}$ and $J(A) = \{x \in A | (e_n^*(xy)) \in c_0 \text{ for all } y \in A\}$ with natural topologies τ_p^A defined by the seminorms $p_p^A(x) = \|e_n^*(xy)\|_p$, $y \in A$. Note that $J(A) \subset K_\infty(A)$ and $J(A)$ carries the relative topology. As in [9], each $K_p(A)$ and $J(A)$ are dense ideals in A satisfying, for $1 \leq p \leq \infty$, $K_1(A) \subset K_p(A) \subset J(A) \subset K_\infty(A)$. We show that if A is a B_0 -algebra with identity having (e_n) unconditional, then A is nuclear iff $K_1(A) = J(A) = K_\infty(A)$ topologically as well.

In §5, we have briefly discussed a number of examples of topological algebras with orthogonal basis, to impress upon the usefulness of algebra point of view in the theory of sequence spaces and to illustrate the general theory. The paper ends with concluding remarks, wherein we discuss three aspects of topological algebras with basis viz the problem of automatic continuity of multiplicative functionals, uniqueness of topology and non-existence of orthogonal basis in an abstract nuclear B_0 -algebra.

By a (locally convex) topological algebra A is meant a Hausdorff (locally convex) topological vector space with topology t which is also a linear associative algebra over complex scalars such that the multiplication is separately continuous, i.e. for each x , the linear maps $L_x y = xy$ $R_x y = yx$ are continuous. A basis (e_n) in A is

orthogonal if $e_n e_m = \delta_{nm} e_m$ for all n, m . For $x \in A$, the expansion $x = \sum_1^\infty \alpha_n e_n$ leads to coefficient functionals $e_n^*(x) = \alpha_n$ which are multiplicative if (e_n) is orthogonal. A nuclear algebra is a topological algebra that is a nuclear locally convex space. A topological algebra with an orthogonal basis is always commutative [8]. We refer to [14], [21] for topological algebras; [15], [19] for locally convex spaces; [12], [13], [16] for sequence spaces; [15] for linear topological aspects of basis theory; and [7], [8], [9], [10] for orthogonal basis. ℓ^p ($1 \leq p \leq \infty$), c, c_0 denote the usual Banach sequence spaces.

2. $\lambda_1(P)$ and topological algebras with basis

Defining property (v) of G_1 -condition implies that $\lambda_1(P)$ is closed under pointwise multiplication and $(\lambda_1(P), \tau_p)$ is a locally convex $*$ -algebra with jointly continuous multiplication having $(\delta_{nm})_{n=0}^\infty$ as an orthogonal Schauder basis. Further, it is easily seen that $\lambda_1(P)$ admits identity (which has to be constant sequence $\{1\}$ denoted by 1) iff $(\lambda_1(P), \tau_p)$ is nuclear; in which case, it is a generalized B^* -algebra in the sense of [1], [3] containing the C^* -sequence algebra $(\ell^\infty, \|\cdot\|_\infty)$ continuously and densely embedded in it. In this section, we aim to incorporate this observation in general theory of topological algebras with basis.

We begin with the following ‘automatic continuity’ result that improves [8, Prop. 3.1] and [5, Th. 1.1] pushing them in the final form leading to a variant in a more general framework. Our arguments are modifications of those in [5] wherein part (i) of the following is proved under the additional assumption that multiplication in A is jointly continuous. This is a strong requirement for non- m -convex non-metrizable algebras.

Theorem 2.1. *If A is a topological vectorspace that is an algebra;*

- (i) *If A is a topological algebra, then every orthogonal basis in A is Schauder.*
- (ii) *If multiplication in A is sequentially separately continuous (i.e. for a sequence (x_n) in A , $x_n \rightarrow 0$ implies $x_n y \rightarrow 0$, $y x_n \rightarrow 0$ for all y), then every orthogonal basis (e_n) in A is b -Schauder (i.e. each coefficient functional e_n^* is bounded in the sense that it maps a bounded set to a bounded set).*

Proof. Let (e_n) be an orthogonal basis in A . Let $n \in \mathbb{N}$ be fixed orthogonality of (e_n) applied to the expansion $x = \sum_1^\infty e_n^*(x) e_n$ implies that $e_n x = e_n^*(x) e_n = x e_n$ for all $x \in A$. Choose a balanced o -neighbourhood U such that $e_n \notin U$. Let $r = \inf \{d > 0 | e_n \in dU\}$. Then $r > 1$.

(i) Let (x_α) be a net in A such that $\lim_\alpha x_\alpha = 0$. Hence $\lim_\alpha x_\alpha e_n = 0$. Given $\delta > 0$, there exists an α_0 such that $e_n^*(x_\alpha) e_n = x_\alpha e_n \in \delta U$ for all $\alpha \geq \alpha_0$. As U is balanced, $|e_n^*(x_\alpha)| e_n \in \delta U$ for $\alpha \geq \alpha_0$. Hence by definition of r , $|e_n^*(x_\alpha)|^{-1} \delta \geq r > 1$, and so $|e_n^*(x_\alpha)| < \delta$ for all $\alpha \geq \alpha_0$. Thus $\lim_\alpha e_n^*(x_\alpha) = 0$.

(ii) It is sufficient to show that e_n^* maps a bounded sequence (x_k) to a bounded sequence. For any sequence $r_k \rightarrow \infty$, $r_k > 0$, $x_k/r_k \rightarrow 0$. Hence $e_n x_k/r_k \rightarrow 0$. Thus $(e_n^*(x_k) e_n)_{k=1}^\infty$ is bounded and for all k , $e_n^*(x_k) e_n \in \lambda U$ for some $\lambda = \lambda(n, U) > 0$. Again by definition of r , $|e_n^*(x_k)| \leq r/\lambda$ for all k . This completes the proof.

It follows, from the continuity of the injective map $x \in A \rightarrow (e_n^*(x)) \in \omega$ (pointwise convergence) that a topological algebra A with an orthogonal basis carries an lmc

topology coarser than the given topology. This makes spectral theory of A tractable; e.g. (e_n^*) constitute the Gelfand space $\chi(A)$ of A viz, the space of all continuous multiplicative functionals on A with weak $*$ topology exactly as in [5, Corollary 1.3] in view of Theorem 2.1; and if A is also a division algebra, then A is isomorphic to scalar field $\mathbb{C}$ (Gelfand-Mazur Theorem). The following consequence of Theorem 2.1 can be regarded as a first step towards classifying topological algebras with orthogonal basis.

COROLLARY 2.2.

Let A and B be topological algebras with orthogonal bases (e_n) and (f_n) respectively.

(a) If $\phi: A \rightarrow B$ is a homeomorphic isomorphism of A onto B , then there exists a permutation σ of $\mathbb{N}$ (the set of natural numbers) satisfying

(C_1) : “ (e_n) and $(f_{\sigma^{-1}(n)})$ are equivalent (in the sense that for any sequence of scalars (x_n) , $\sum x_n e_n$ converges in A iff $\sum x_n f_{\sigma^{-1}(n)}$ converges in B ”,

and ϕ is of the form $\phi = T_\sigma$ where $T_\sigma(\sum x_n e_n) = \sum x_{\sigma(n)} f_n$.

(b) For any permutation σ of $\mathbb{N}$ satisfying (C_1) , T_σ defines an isomorphism of A onto B . Further, if A and B are F -algebras (i.e. complete metric algebras), then T_σ is a homeomorphism.

Proof. (a) By Theorem 2.1, (e_n) and (f_n) are Schauder bases, and as noted above $\chi(A) \approx \mathbb{N} \approx \chi(B)$. The conjugate map between the topological duals $\phi^*: B^* \rightarrow A^*$, $\phi^* f = f \circ \phi$ is a bijective linear map that is $\sigma(A^*, A) - \sigma(B^*, B)$ continuous. Further, ϕ being an isomorphism, $\phi^*(\chi(B)) \subset \chi(A)$. Hence there is a permutation $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that $\phi^*(f_n^*) = e_{\sigma(n)}^*$. Thus for all x in A , $f_n^*(\phi(x)) = e_{\sigma(n)}^*(x) = x_{\sigma(n)}$; and so $\phi(\sum x_n e_n) = \sum x_{\sigma(n)} f_n = \sum x_n f_{\sigma^{-1}(n)}$ giving (a).

(b) Let σ be a permutation of $\mathbb{N}$ satisfying (C_1) . Then for any $x = \sum x_n e_n$ in A , $\sum x_n f_{\sigma^{-1}(n)} = \sum x_{\sigma(n)} f_n$ converges in B , thus defining $\phi(x)$ in B . This defines a linear map $\phi: A \rightarrow B$ which preserves the ring structure, as (e_n) and (f_n) are orthogonal. Condition (C_1) also implies that ϕ is bijective. By Theorem 2.1, the linear maps

$$T_n: A \rightarrow B, T_n(x) = \sum_{k=1}^n e_k^*(x) f_{\sigma^{-1}(k)}$$

are continuous, and $\phi(x) = \lim_{n \rightarrow \infty} T_n x$ (pointwise) for all $x \in A$. Hence if A and B are F -algebras, then ϕ is continuous; and hence is a homeomorphism by open mapping theorem.

Remarks 2.3. (a) In the absence of metrizability, condition (C_1) is not sufficient to make T_σ a homeomorphism. This is seen in the case of $A = (\ell^1, \|\cdot\|_1)$, $B = (\ell^1, \eta(\ell^1, \ell^\infty))$; both are topological algebras with $e_n = (\delta_{nm})$ as orthogonal basis. Here the normal topology $\eta(\ell^1, \ell^\infty)$ on ℓ^1 is defined by seminorms $p_x(y) = \sum |y_i x_i|$, $x \in \ell^\infty$.

(b) Now suppose A and B are locally convex. Let $S(A)$ (resp. $S(B)$) be a family of continuous seminorms determining the topology of A (resp. B). Given a homeomorphic isomorphism $\phi: A \rightarrow B$, let σ be the permutation of $\mathbb{N}$ defined by Corollary 2.2 (a) so

that $\phi = T_\sigma$. A consequence of continuity of ϕ and that of ϕ^{-1} is that σ satisfies

- (C₂): (i) Given $q \in S(B)$, there exists $ap \in S(A)$ and $ak > 0$ such that $q(f_{\sigma^{-1}(n)}) \leq kp(e_n)$ for all n .
 (ii) Given $ap \in S(A)$, there exists $aq \in S(B)$ and $ak' > 0$ such that $p(e_n) \leq k'q(f_{\sigma^{-1}(n)})$ for all n .

The following shows that this suffices in the nuclear case. Note that in a B_0 -space, a Schauder basis is equicontinuous [15, Th. 10.1.2].

PROPOSITION.

Let A and B be nuclear locally convex algebras with equicontinuous orthogonal bases (e_n) and (f_n) respectively. The isomorphism $T_\sigma: A \rightarrow B$ defined by a permutation σ of $\mathbb{N}$ satisfying (C₁) and (C₂) is a homeomorphism.

Indeed, due to nuclearity, (e_n) is an absolute basis [15, Theorem 10.2.1] so that given $p \in S(A)$, there is a $p' \in S(A)$ such that $\Sigma e_n^*(x)p(e_n) \leq p'(x)$ for all x . Hence for $aq \in S(B)$, (C₂) (i) gives, for $x \in A$,

$$q(T_n x) \leq \sum_{i \in I} e_i^*(x) q(f_{\sigma^{-1}(i)}) \leq k \sum_{i=1}^n e_i^*(x) p(e_i) \leq kp'(x).$$

Thus (T_n) is an equicontinuous family. This makes ϕ continuous. Similarly C₂ (ii) gives continuity of ϕ^{-1} .

Now it is well known [15, Th. 10.1.4] that a complete locally convex space with an absolute basis is linearly homeomorphic to a Köthe space $(\lambda(P), \tau_P)$. Further, by the Basis Theorem [15, Th. 10.2.1], an equicontinuous basis in a nuclear space is absolute, with the result, a complete nuclear space with an equicontinuous basis has to be a Köthe space. The following supplements this by exploiting the added ring theoretic structure on $\lambda_1(P)$. We shall call a power set P with property (v) of §1 semi- G_1 -set and $\lambda_1^0(P)$ denotes a semi- G_1 -Köthe space.

Theorem 2.4. Let (A, t) be a topological algebra satisfying the following

- (i) A is locally convex
 (ii) A has jointly continuous multiplication
 (iii) A admits an orthogonal absolute basis (e_n) . Then there exists a homeomorphic isomorphism $\phi: (A, t) \rightarrow (\lambda_1^0(P), \tau_P)$ from A onto a dense subalgebra of some $\lambda_1^0(P)$. Further, (a) if A is a $*$ -algebra with continuous involution and (e_n) is hermitian (i) (i.e. each e_n a hermitian element), then $\phi(A)$ is a $*$ -subalgebra of $\lambda_1^0(P)$ and ϕ is a $*$ -isomorphism; (b) if A is sequentially complete, then ϕ is onto.

Proof. By (ii), given a o -neighbourhood U , there exists a o -neighbourhood V such that $VV \subset U$. Additionally using (i), the topology of A is determined by a family $\Gamma = \{p_\alpha | \alpha \in \Delta\}$ of seminorms satisfying: For each $\alpha \in \Delta$, there exists a $\beta \in \Delta$ such that

$$p_\alpha(xy) \leq p_\beta(x)p_\beta(y) \quad (x, y \text{ in } A). \quad (1)$$

Let (e_n) be an orthogonal absolute basis. For each $\alpha \in \Delta$, let $a^{(\alpha)} = (a_n^{(\alpha)}) \in \omega$, $a_n^{(\alpha)} = p_\alpha(e_n)$. By (2) and Hausdorff property of A , $P = \{a^{(\alpha)} | \alpha \in \Delta\}$ is a power set. The Köthe topology

τ_P on $\lambda(P)$ is determined by $\Gamma_P = \{q_\alpha | \alpha \in \Delta\}$, $q_\alpha(b) = \sum_1^\infty |b_n| a_n^{(\alpha)} < \infty$ for all $\alpha \in \Delta$. Expanding each $x \in A$ as $x = \sum_1^\infty e_n^*(x) e_n$, orthogonality of (e_n) implies that each e_n^* is a multiplicative linear functional, which is continuous by Theorem 2.1. As (e_n) is absolute, for each α , there is a β satisfying, for all x ,

$$\sum_{n=1}^{\infty} |e_n^*(x)| p_\alpha(e_n) \leq p_\beta(x). \quad (2)$$

This with continuity of p_α , implies that $p_\alpha(x) \leq \sum |e_n^*(x)| a_n^{(\alpha)} \leq p_\beta(x) < \infty$ showing $(e_n^*(x)) \in \lambda(P)$. Thus, there is defined linear map $\phi: A \rightarrow \lambda(P)$, $\phi(x) = (e_n^*(x))$ satisfying: Given $\alpha \in \Delta$, there is a $\beta \in \Delta$ such that $p_\alpha(x) \leq q_\alpha(\phi(x)) \leq p_\beta(x)$. Thus ϕ is one-one and is a homeomorphism into. Orthogonality of (e_n) has the following consequences;

1° Taking $x = y = e_n$ in (1), given α , there is a β such that $a_n^{(\alpha)} = p_\alpha(e_n) = p_\alpha(e_n^2) \leq (p_\beta(e_n))^2 = (a_n^{(\beta)})^2$. Thus, P satisfies semi- G_1 -property and $(\lambda(P), \tau_P)$ is a complete locally convex topological algebra.

2° For $x = \sum_1^\infty e_n^*(x) e_n$, $y = \sum_1^\infty e_n^*(y) e_n$; $xy = \sum_{n,k} e_n^*(x) e_k^*(y) e_n e_k = \sum_n e_n^*(x) e_n^*(y) e_n$ giving $\phi(xy) = \phi(x)\phi(y)$.

(a) If $x \rightarrow x^*$ is the continuous involution of A with each e_n hermitian, then for an $x \in A$, $x^* = (\sum e_n^*(x) e_n)^* = \sum \overline{e_n^*(x)} e_n$, $\phi(x^*) = \phi(x)^*$.

(b) Let A be sequentially complete. Let $u = (u_n) \in \lambda(P)$ so that for each α , $\sum_1^\infty p_\alpha(u_n e_n) = \sum_1^\infty |u_n| a_n^{(\alpha)} < \infty$. Hence taking $x^{(n)} = \sum_1^n u_k e_k$, for $n > m$ and for each α , $p_\alpha(x^{(n)} - x^{(m)}) \leq \sum_{m+1}^n |u_k| p_\alpha(e_k) \rightarrow 0$. Thus $(x^{(n)})$ is Cauchy, hence $x^{(n)} \rightarrow x \in A$ and $x = \sum u_n e_n$ showing that ϕ is onto. This completes the proof.

The following is immediate from [18, Th. 2.17] and [15, Th. 10.1.2]. A B_0 -algebra is a complete metrizable locally convex algebra. A locally m -convex algebra [14] is a topological algebra whose topology is determined by a separating family of submultiplicative seminorms.

COROLLARY 2.5.

A nuclear B_0 -algebra with an orthogonal basis is homeomorphically isomorphic to a semi- G_1 Köthe algebra $(\lambda_1^0(P), \tau_P)$ for a countable P .

It is shown in [8, p. 246] that an infinite dimensional normed algebra with orthogonal basis cannot have identity. The following describes some connections between the existence of identity and nuclearity.

COROLLARY 2.6.

Let A be a locally convex algebra having an orthogonal basis (e_n) . Let multiplication in A be jointly continuous.

(1) *Let (e_n) be an absolute basis. If A has identity, then A is nuclear.*

(2) *If A has identity, then (e_n) is bounded.*

(3) *If A is sequentially complete and (e_n) is bounded and equicontinuous, then A has identity.*

Proof. (1) Let e be the identity of A . Then $e = \sum_1^\infty e_n$, $\phi(e) \in \lambda_1^0(P)$ and for each α , $\sum a_n^{(\alpha)} = q_\alpha(\phi(e)) < \infty$. Thus $P \subset \ell^1$, $\ell^\infty \subset \lambda_1^0(P)$. This also implies $\phi(e) = 1$ (constant sequence 1) and by the proof of Theorem 2.4; for each α , there is a β such that

$a_n^{(\alpha)}/a_n^{(\beta)} \leq a_n^{(\beta)}$, $(a_n^{(\beta)}) \in \ell^1$. Hence $(\lambda_1^0(P), \tau_P)$ is nuclear. Therefore $\phi(A)$, and hence A , is nuclear.

(2) $e = \sum e_n$ implies $e_n \rightarrow 0$, hence $(p_\alpha(e_n))$ is bounded for each continuous seminorm p_α on A .

(3) In the notations of proof of Theorem 2.4, for each α there is a scalar $M_\alpha > 0$ satisfying $p_\alpha(e_n) \leq M_\alpha$ for all n . Now A is nuclear, (e_n) is equicontinuous, hence (e_n) is absolute. Thus $\lambda_1^0(P)$ is nuclear. Hence for each α , there is a β such that $p_\alpha(e_n)/M_\beta \leq p_\alpha(e_n)/p_\beta(e_n)$ and $\{p_\alpha(e_n)/p_\beta(e_n)\} \in \ell^1$. Then $P \subset \ell^1$, $\ell^\infty \subset \lambda_1^0(P)$, hence $\lambda_1^0(P)$, and so A has identity.

Example 2.7. Converse of Corollary 2.6 (1) does not hold. Consider the G_∞ -Köthe space $(\lambda_\infty(P), \tau_P)$ assumed nuclear. There is a $d \in P$ such that $\{1/d_n\} \in \ell^1$. Given x, y in $\lambda_\infty(P)$, $p_\alpha(xy) = \sum_1^\infty |x_n y_n| a_n \leq \|1/d\|_1 p_\alpha(x) p_\alpha(y)$ showing $\lambda_\infty(P)$ to be a locally convex algebra having absolute basis $e_n = (\delta_{nm})$ and with jointly continuous multiplication. Now if $1 \in \lambda_\infty(P)$, then $P \subset \ell^1$, hence for all $a \in P$, $a_n \rightarrow 0$ contradicting $a_n \leq a_{n+1}$ for all n . Thus $\lambda_\infty(P)$ cannot have identity. In fact, $\lambda_\infty(P) \subset \ell^1$. Taking P countable $\lambda_\infty(P)$ can be chosen a B_0 -algebra. Thus, in view of Corollary 2.6. (3), in a nuclear B_0 -algebra, (e_n) need not be automatically bounded, though it is equicontinuous [15, Th. 10.1.2]; and nuclear B_0 -algebra with an orthogonal basis need not be unital.

Example 2.8. In Corollary 2.6 (1), the hypothesis that (e_n) is absolute cannot be omitted even if A is assumed metrizable. Let $A = \{x = (x_n) \in \omega \mid \lim_n x_n n^{-1/2^k} = 0 \text{ for all } k = 1, 2, 3, \dots\}$. A is a B_0 -algebra with topology defined by the seminorms $p_k(x) = \sup_n |x_n| n^{-1/2^k}$ having identity 1 and admitting orthogonal unconditional basis $e_n = (\delta_{nm})$. The algebra A is not nuclear (in view of Theorem 4.2, since $K(A) \neq J(A)$) as shown in [9] and (e_n) fails to be an absolute basis (because, otherwise Theorem 2.2 applies) though it is equicontinuous.

Similarly, the hypothesis that A has identity can also not be omitted. Consider for a G_1 -power set P , the Köthe sequence algebra $(\lambda_1(P), \tau_P)$. Then $e_n = (\delta_{nm})$ is an absolute basis, and $\lambda_1(P)$ is nuclear iff $\lambda_1(P)$ has identity.

COROLLARY 2.9. [8].

Let A be a complete locally m -convex algebra with an orthogonal basis (e_n) . If A has identity e , then A is metrizable and is homeomorphically isomorphic to (ω, t) , t being the topology of pointwise convergence.

Proof. By [8, Th. 1.3], an lmc algebra with an orthogonal basis having identity is metrizable. Thus (e_n) is equicontinuous. If $\Gamma = (p_k)_{k=1}^\infty$ is a sequence of submultiplicative seminorms determine the topology on A , then $p_k(e_n) \leq p_k(e_n)^2$ gives $p_k(e_n) \geq 1$ or $p_k(e_n) = 0$. But $e = \sum_{n=1}^\infty e_n$ implies $p_k(e_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, for each k , $p_k(e_n) = 0$ for all $n \geq$ some N_k . This makes (e_n) an absolute basis. By Theorem 2.4, $A \approx \lambda_1^0(P)$ and $P = \varphi$, the space of sequences with only finitely many non-zero terms. From this, the assertion follows immediately.

The following two particular cases of Köthe algebra $\lambda(P)$ reveal the importance of the existence of identity in the above Corollary.

Example 2.10. In the absence of identity, an lmc algebra with an orthogonal basis

need not be metrizable. Consider the sequence algebra $\varphi = \{x \in \omega \mid x_n \neq 0 \text{ only for finitely many } n\}$. Let $\varphi_m = \{x \in \varphi \mid x_k = 0 \text{ for } k \geq m + 1\}$, a finite dimensional ideal in φ for each m , and $\varphi = \bigcup_{m=1}^{\infty} \varphi_m$. Let τ be the strongest locally convex linear topology on φ making each of the embeddings $\varphi_m \rightarrow \varphi$ continuous. It follows from [20, pp. 195–196] that (φ, τ) is an lmc algebra and τ is the finest lmc topology making each of above embeddings continuous. Also, $e_n = (\delta_{nm})$ is an orthogonal basis. A similar phenomena is also exhibited by the algebra $A = \bigcup_{1 \leq p < \infty} \ell^p$, each ℓ^p carries the relative topology from ℓ^∞ . The topology τ is not metrizable.

Example 2.11. An lmc algebra with an orthogonal basis, even if non-normed Frechet, need not be the whole of ω . This is exhibited by the sequence algebra $B = \bigcap_{1 < p \leq \infty} \ell^p$ with the topology defined by $\|\cdot\|_p$: $p = 1, 2, 3, \dots, \infty$. It is shown in [5, Th. 2.1] that an lmc algebra with orthogonal basis and having identity carries via the mapping φ of Theorem 2.3, the topology of pointwise convergence. This example, as well as the algebra $L^\omega(T)$ discussed in §5.5 show that this need not be the case in the absence of identity.

Remark 2.12. There is another interesting characterization of ω . A σ - C^* -algebra is complete metrizable lmc algebra that is an inverse limit of C^* -algebras.

PROPOSITION [2]

A commutative σ - C^* -algebra with identity that is a nuclear space admits an orthogonal basis and is homeomorphically isomorphic to ω .

The following was proved for Banach algebras in [9]. It reveals ‘sequential nature’ of a topological algebra with an orthogonal basis.

Theorem 2.13. *Let (A, t) be a locally convex algebra with an orthogonal basis (e_n) . Consider the following*

- (i) *A is sequentially complete with jointly continuous multiplication and (e_n) is absolute and bounded*
- (ii) *A is locally m -convex*
- (iii) *A is a B_0 -algebra.*

Let A satisfy at least one of the above conditions. Then A is a countable direct sum of simple topological algebras Ae_n , each of which is homeomorphically isomorphic to $\mathbb{Q}$.

Proof. As in [9, Th. 4.4, Prop. 4.5], each Ae_i is a closed minimal ideal, $A = \sum_1^\infty Ae_i$, each $M_n = \ker e_n^*$ is a closed maximal ideal and for each n , $\psi: (A/M_n, t_q) \rightarrow (Ae_n, t)$, $\psi(x + M_n) = xe_n$ is a continuous isomorphism from A/M_n onto Ae_n . Here t_q = quotient topology on A/M_n and t = relative topology from A on Ae_n .

(a) Assume (i). In notations of proof of Theorem 2.4, consider the homeomorphic isomorphism $\phi: (A, t) \rightarrow (\lambda_1^0(P), \tau_P)$. Since (e_n) is bounded, $P \subset \ell^\infty$, hence $\ell^1 \subset \lambda_1^0(P)$, in fact, ℓ^1 is sequentially dense in $\lambda_1^0(P)$. Then the subalgebra $B = \phi^{-1}(\ell^1)$ of A , which is a Banach algebra with norm $\|x\| = \|\phi(x)\|_1$, is sequentially dense in (A, t) . By Theorem 2.1, $B \cong B_0(\mathbb{Q}, \mathbb{Q})$. By Proposition 2.13, B is a countable direct sum of simple topological algebras $B e_n$, each of which is homeomorphically isomorphic to $\mathbb{Q}$.

$B/(M_n \cap B)$ is sequentially t_q -dense in A/M_n . Thus A/M_n is $\mathcal{C}$; $t_q = t$ via ψ and the conclusion follows.

(b) Assume (ii) or (iii). $(A/M_n, t_q)$ is a division algebra that is either a B_0 -algebra or is lmc. Since Gelfand-Mazur Theorem holds in those two classes of algebras ([14], [21]); $(A/M_n, t_q)$, hence (Ae_n, t) , is $\mathcal{C}$. This completes the proof.

3. Abstract characterizations of ℓ^∞ and c_0

A dual Banach algebra $(A, \|\cdot\|)$ is a Banach algebra that is dual, as a Banach space, of some Banach space $(A_*, \|\cdot\|)$ called its predual. A dual algebra isomorphism $\psi: A \rightarrow B$ from a dual Banach algebra A to a dual Banach algebra B is an algebra isomorphism that is $\|\cdot\| - \|\cdot\|$ homeomorphism as well as $\sigma(A, A_*) - \sigma(B, B_*)$ homeomorphism. The Banach algebra $\ell^\infty (= \lambda(P))$ with $P = \{a \in \ell^1/a_n \geq 0 \text{ for all } n\}$ is a dual Banach algebra with 1 having predual $(\ell^1, \|\cdot\|_1)$. Further, $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$ is a sequentially complete topological algebra having orthogonal basis $e_n = (\delta_{nm})_1^\infty$.

Theorem 3.1. Let A be a dual Banach algebra with identity e and predual A_* . If (i) $(A, \sigma(A, A_*))$ is sequentially complete, and (ii) $(A, \sigma(A, A_*))$ admits an equicontinuous orthogonal basis (e_n) , then there exists a dual algebra isomorphism ϕ from A onto ℓ^∞ .

Proof. Define $\phi: A \rightarrow \omega$ as $\phi(x) = (e_n^*(x))$. Each e_n^* being multiplicative, it is $\|\cdot\|$ continuous, and for all x , $|e_n^*(x)| \leq \|x\|$. Thus $\phi(x) \in \ell^\infty$, $\phi(A) \subset \ell^\infty$ and $\|\phi(x)\|_\infty \leq \|x\|$. As in the proof of Theorem 2.4, ϕ is one-one, linear and is homeomorphism.

We show that ϕ is $\sigma(A, A_*) - \sigma(\ell^\infty, \ell^1)$ continuous. Let $g = (g_n) \in \ell^1$, regarded as a linear functional on ℓ^∞ . Thus $g \circ \phi \in A_*$. Indeed, for all $x \in A$ $|(g \circ \phi)(x)| = |\sum \langle g, \phi(x) \rangle_n|$, $g_n > 0 \leq \sum |g_n| |e_n^*(x)|$. By equicontinuity of (e_n) in $\sigma(A, A_*)$, given $\varepsilon > 0$, there is a σ -neighbourhood U in $\sigma(A, A_*)$ satisfying, for all n , $|e_n^*(x)| \leq \varepsilon$. Thus for all $x \in U$, $|(g \circ \phi)(x)| \leq \varepsilon \|g\|_1$, giving $g \circ \phi \in A_*$. Thus if $x \rightarrow 0$ in $\sigma(A, A_*)$ (so that $f(x) \rightarrow 0$ for all $f \in A_*$), $g(\phi(x)) \rightarrow 0$ for all $g \in \ell^1$ giving $\phi(x) \rightarrow 0$ in $\sigma(\ell^\infty, \ell^1)$.

Now $e = \sum_1^\infty e_n$ in $\sigma(A, A_*)$. Hence for all $f \in A_*$, $(f(e_n) \in \ell^1$ and $f(1) = \sum_1^\infty f(e_n)$. This we use to show that ϕ is onto ℓ^∞ . Let $y \in \ell^\infty$, $y = (y_n)$. Let $e'_n = (\delta_{nm})$, $y^{(k)} = \sum_1^k y_n e'_n$. Then for all $f \in A_*$, $k \geq m$, $|f(x^{(k)} - x^{(m)})| \leq |\sum_{m+1}^\infty |y_n| |f(e_n)| \leq \|y\|_\infty |\sum_{m+1}^k f(e_n)| \rightarrow 0$ as $k, m \rightarrow \infty$. Thus $(x^{(k)})$ is $\sigma(A, A_*)$ Cauchy, hence $x^{(k)} \rightarrow x$ in A in $\sigma(A, A_*)$. Therefore $\phi(x) = \lim_k \phi(x^{(k)})$ (in $\sigma(\ell^\infty, \ell^1)$) = y .

It follows, by the open mapping theorem, that ϕ is $\|\cdot\| - \|\cdot\|_\infty$ homeomorphism. To show that ϕ is $\sigma(\ell^\infty, \ell^1) - \sigma(A, A_*)$ continuous, it suffices to show $\phi^*(\ell^1) = A_*$, $\phi^*: (\ell^\infty)^* \rightarrow A^*$ being the conjugate of ϕ between the respective norm duals, $\phi^*(g) = g \circ \phi$. We have already shown $\phi^*(\ell^1) \subset A_*$. Let $f \in A_*$, let $y_n = f(e_n)$. Then $y = (y_n) \in \ell^1$ and for all n , $\langle f, e_n \rangle = y_n = \langle y_n, e'_n \rangle = \langle y, \phi(e_n) \rangle = \langle \phi^*(y), e_n \rangle$, and so, for all $x \in A$, $\langle f, x \rangle = \langle f, \sum_1^\infty x_n e_n \rangle = \sum_1^\infty x_n \langle f, e_n \rangle = \sum_1^\infty x_n y_n = \langle \phi^*(y), x \rangle$. Thus $f = \phi^*(y) \in \phi^*(\ell^1)$ implying $\phi^*(\ell^1) = A_*$. This completes the proof.

A uB -algebra (uniform Banach algebra) is a Banach algebra $(A, \|\cdot\|)$ with $\|x^2\| = \|x\|^2$ for all $x \in A$.

Proof. If $r(\cdot)$ denotes spectral radius of an element, then for all $x \in A$, $\|x\| = \limsup_{n \rightarrow \infty} \|x^n\|^{1/n} = r(x) = r(\phi(x))$ (by Theorem 3.1) $= \|\phi(x)\|_\infty$, ℓ^∞ being a C^* -algebra.

Theorem 3.3. *A uB -algebra $(A, \|\cdot\|)$ with an orthogonal basis (e_n) is isometrically isomorphic to the Banach algebra c_0 .*

Proof. A cannot have identity by [8, p. 346], and it is well known that the uniform condition on A forces A to be commutative. Gelfand theory provides a homomorphism $\phi: A \rightarrow C_0(\chi(A))$ of A into a subalgebra of the supnorm Banach algebra $C_0(\chi(A))$ of continuous functions on the (locally compact) Gelfand space $\chi(A)$ vanishing at infinity. Further, by the uniform condition, the Gelfand map ϕ is an isometry. Thus ϕ is a closed subalgebra of $C_0(\chi(A))$. But by [8], $\chi(A) = \mathbb{N}$. Hence $C_0(\chi(A)) = c_0$. As A has an orthogonal basis, the sequence algebra ϕ (sequences with only finitely many non-zero terms), which is already dense in c_0 , is dense in $\phi(A)$. Thus $\phi(A) = c_0$, and the proof is complete.

A Hilbert algebra A with a biorthonormal basis (e_n) is a Hilbert space A which is also a Banach algebra with orthogonal basis (e_n) such that (e_n) is an orthonormal family. It can be shown, as above, that a Hilbert algebra with a biorthonormal basis is isometrically isomorphic to ℓ^2 .

4. A characterization of nuclearity: Abstract Grothendieck-Pietsch criterion

DEFINITION 4.1.

Let A be a topological algebra with an orthogonal basis (e_n) .

- (a) (i) For $1 \leq p < \infty$, let $K_p(A) = \{x \in A | (e_n^*(xy)) \in \ell^p \text{ for all } y \in A\}$; (ii) $J(A) = \{x \in A | e_n^*(xy) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } y \in A\}$; (iii) $J_c(A) = \{x \in A | e_n^*(xy) \text{ is convergent for all } y \in A\}$.
 (b) The natural topologies on these subsets of A are defined as follows (i) For $1 \leq p < \infty$, τ_A^p is the locally convex topology on $K_p(A)$ defined by semi-norms $\{p_y^p | y \in A\}$, $p_y^p(x) = (\sum |e_n^*(xy)|^p)^{1/p}$.

- On $K_\infty(A)$, the topology τ_A^∞ is defined by seminorms $p_y^\infty(x) = \sup |e_n^*(xy)|$, $y \in A$;
 (ii) On $J(A)$ and $J_c(A)$, we consider the relative topology from $(K_\infty(A), \tau_A^\infty)$.

The sets $K_1(A)$ and $J(A)$ are considered in [9] wherein the problem of equality $K_1(A) = J(A)$ is discussed. The following is essentially due to [9]. Our presentation clarifies configurations of $K_p(A)$, $J_c(A)$ and $J(A)$ in A ; also part (e) refines [9, Prop. 6].

PROPOSITION 4.1.

Let A be a topological algebra with an orthogonal basis (e_n) .

- (a) Each of $K_p(A)$, $1 \leq p < \infty$, $J(A)$ and $K_\infty(A)$ is an ideal in A .
 (b) $K_1(A)$ is the intersection of dense maximal ideals in A if (e_n) is unconditional
 (c) For $1 \leq p < \infty$, $\overline{K_p(A)} = \overline{J(A)} = \overline{J_c(A)} = A$
 (d) Let A have identity e and (e_n) be unconditional. Via the mapping $\phi: x \in A \rightarrow (e_n^*(x)) \in \omega$ which is one-one, $K_p(A)$ is a subalgebra of ℓ^p ($1 \leq p \leq \infty$), $J_c(A)$ is a subalgebra of c_0 and $J(A)$ is a subalgebra of ℓ^ω . The horizontal and vertical single arrows in the following define continuous embedding when the spaces involved carry their natural

topologies.

$$\begin{array}{ccccccc}
 \ell^1 & \xrightarrow{id} & \ell^p & \longrightarrow & c_0 & \longrightarrow & c \longrightarrow \ell^\infty \cdots \\
 \phi \uparrow & & \phi \uparrow & & \phi \uparrow & & \phi \uparrow & & \phi \uparrow & & \phi^{-1} \\
 K_1(A) & \xrightarrow{id} & K_p(A) & \longrightarrow & J(A) & \longrightarrow & J_c(A) & \longrightarrow & K_\infty(A) & \cdots & \longrightarrow A
 \end{array}$$

(e) Let A have identity e .

(i) If there exists an invertible element x in A such that $(e_n^*(x)) \in \ell^p$ ($1 \leq p < \infty$), then $K_p(A) = J(A) = J_c(A) = K_\infty(A)$.

(ii) If there exists an invertible element x in A such that $e_n^*(x) \rightarrow 0$, then $J(A) = J_c(A) = K_\infty(A)$.

(f) Let A be a B_0 -algebra. Then its dual A^* is linearly homeomorphic to $K_1(A)$.

The algebra A of Example 2.8, as shown in [9], is such that $K_1(A) \neq J(A)$. Incidentally, this A is not nuclear. On the other hand, it is also noted in [9] that for the Frechet algebras ω and $H(D)$ (holomorphic functions on open unit disc with Hadamard product—see Example 5.7) [8], $K_1(A) = J(A)$; and both ω and $H(D)$ are nuclear. The following explains this.

Theorem 4.2. Let A be a B_0 -algebra with identity e and having an orthogonal basis (e_n) .

(a) If A is nuclear, then for $1 \leq p < \infty$, $K_1(A) = K_p(A) = J(A) = J_c(A) = K_\infty(A)$ algebraically as well as topologically.

(b) Let (e_n) be unconditional, if $K_1(A) = J(A)$ algebraically as well as topologically, then A is nuclear.

Proof. A being B_0 , multiplication in A is jointly continuous and (e_n) is equicontinuous. The nuclearity implies [15, Th. 10.21] that (e_n) is absolute. By Theorem 2.4, $\phi(x) = (e_n^*(x))$ establishes a homeomorphic isomorphism from A onto the semi- G_1 -Köthe algebra $\lambda_1^0(P)$ (denoted by $\lambda_1(P)$ only) with $P = \{(p_\alpha(e_n)) | \alpha \in \mathbb{N}\}$, $\Gamma = \{p_\alpha | \alpha \in \mathbb{N}\}$ being a countable family of seminorms determining the topology on A . Further, since A has identity, $P \subset \ell^1$ and so $\lambda_1(P)^\times \subset \ell^1 \subset \ell^\infty \subset \lambda_1(P)$. Since each e_n^* is multiplicative, $K_1(A) = \{x \in A | \sum |e_n^*(x)| |e_n^*(y)| < \infty \text{ for all } y \in A\}$. This, with $\phi(A) = \lambda_1(P)$ implies that $K_1(A) \subset \phi^{-1}(\lambda_1(P))$. We claim that $\phi(K_1(A)) = \lambda_1(P)^\times$. Indeed, let $b = (b_n) \in \lambda_1(P)^\times$, $\sum |b_n| |e_n^*(y)| < \infty$ for all $y \in A$. Then $b \in \ell^1 \subset \ell^\infty$. As (e_n) is absolute, it is unconditional, hence a bounded multiplier basis, with the result, as noted in [17, Lemma 1], given $c = (c_n) \in \ell^\infty$, there exists an $x \in A$ such that $x = \sum_1^\infty c_n x_n$, hence $e_n^*(x) = c_n$ for all n . Thus, $b = \phi(x)$ for some $x \in A$. This implies $x \in K_1(A)$ and $(K_1(A), \tau_A^1)$ is homeomorphically isomorphic to $(\lambda_1(P)^\times, \tau_P^\times)$.

Now as A is nuclear and B_0 , $(\lambda_1(P), \tau_P)$ is also nuclear and B_0 . But then, as is well known, $(\lambda_1(P)^\times, \tau_P^\times)$, and hence $(K_1(A), \tau_A^1)$ is also nuclear. Therefore by the Grothendieck-Pietsch nuclearity criteria the topology τ_A^1 on $K_1(A)$ is also determined by the seminorms $p_y^\infty(x) = \sup \{|e_n^*(x)| |e_n^*(y)|\}$, $y \in A$. Thus via ϕ and ϕ^{-1} , $K_1(A) = \{u = (u_n) \in \omega | (|u_n| |e_n^*(x)|) \in \ell^\infty\} \supset K_\infty(A) \supset J(A) \supset K_1(A)$. This proves (a).

(b) Assume that $K_1(A) = J(A)$ topologically also. Since A has identity $e = \sum_1^\infty e_n$, we have $(p_\alpha(e_n)) \in c_0$ for all α . As (e_n) is unconditional, there is a $z \in A$ such that $e_n^*(z) = p_\alpha(e_n)$.

Further for any $x \in A$, $x = \sum_1^\infty e_n^*(x) e_n$ implies that $|e_n^*(xz)| = |e_n^*(x)| p_\alpha(e_n) = p_\alpha(e_n^*(x) e_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus $(p_\alpha(e_n)) = (e_n^*(z)) \in J(A) = K_1(A) \subset \ell^1$ via ϕ by Proposition 4.1. Now since (e_n) is equicontinuous, given α , there is a β and a constant $k = k_\alpha$ such that $|e_n^*(x)| p_\alpha(e_n) \leq k p_\beta(x)$ for all $x \in A$ [15, p. 172]. This gives, for all $x \in A$, $\sum |e_n^*(x)| p_\alpha(e_n) \leq k p_\beta(x) \sum p_\alpha(e_n) = k M p_\beta(x)$ (say) showing that (e_n) is absolute. It follows, by Theorem 2.1, that ϕ establishes a homeomorphic isomorphism between A and $\lambda_1(P)$, $P = \{(p_\alpha(e_n)) | \alpha \in N\}$. But then $(K_1(A), \tau_A^1)$ is homeomorphically isomorphic to $(\lambda_1(P))^* \cdot \tau_P^*$. Now the assumption implies that τ_A^1 is also determined by the seminorms $\{p_\nu^* | \nu \in A\}$. It follows that $(K_1(A), \tau_A^1)$ and so $(\lambda_1(P))^* \cdot \tau_P^*$, is nuclear. Metrizability of $\lambda_1(P)$ implies that $(\lambda_1(P), \tau_P)$, and so A , is nuclear. This completes the proof.

5. Examples

In this section, we point out several examples of topological algebras with basis. Our point of view is that almost all sequence spaces discussed in the literature are sequence algebras, which via Theorem 2.4, provide framework for various convolution algebras of functions and distributions. Throughout, P denotes a power set. We omit easy proofs.

(5.1) Consider the cross-dual $\lambda(P)^* = \{x \in \omega | (x_n y_n) \in \ell^1 \text{ for all } y \in \lambda(P)\}$ with the cross-dual topology τ_P^* determined by seminorms $\{p_x | x \in \lambda(P)\}$, $p_x(y) = \|xy\|_1$. If $\lambda_1(P)$ is nuclear, then $(\lambda_1(P)^* \cdot \tau_P^*)$ is a topological $*$ -algebra without identity with orthogonal basis $e_n = (\delta_{nm})$ and $\lambda_1(P)^* \subset \ell^\infty$. If $\lambda_\infty(P)$ is nuclear, then $(\lambda_\infty(P)^* \cdot \tau_P^*)$ is also a topological $*$ -algebra with identity containing ℓ^∞ as a dense $*$ -subalgebra—in fact, it is a GB^* -algebra. By [4, p. 91], for a barrelled $\lambda(P)$, the dual $\lambda(P)^* = \lambda(P)^* = \{x \in \omega | \text{there exists } a \in P \text{ and } 0 < \rho_x < \infty \text{ with } |x_n| \leq \rho_x a_n \text{ for all } n\} = \cup \{N_a | a \in P\}$, where $N_a = \{x \in \lambda(P)^* | \sup(|x_n|/a_n) < \infty\}$, is a Banach space with norm $\rho_a(x) = \inf\{\lambda > 0 | |x_n| \leq \lambda a_n \text{ for all } n\}$. Let ρ be the strongest locally convex linear topology making each embedding $id_a: (N_a, \rho_a) \rightarrow \lambda(P)^*$ continuous. For a nuclear $\lambda_1(P)$, N_a turns out to be an ideal; and for a with $a_n \leq 1$ for all n , (N_a, ρ_a) is a Banach $*$ -algebra. Using [20], other useful topologies can also be defined on $\lambda(P)^*$.

(5.2) We consider non-nuclear variants of Köthe sequence algebras. For $1 \leq p \leq \infty$, let $\ell^p(P) = \{x \in \omega | (x_n a_n) \in \ell^p \text{ for all } a \in P\}$ with topology τ_P^p defined by semi-norms $\{p_a^p | a \in P\}$, $p_a^p(x) = \|xa\|_p$. Then $(\ell^p(P), \tau_P^p)$ is a complete topological $*$ -algebra with jointly continuous multiplication; and for $1 \leq p < \infty$, $e_n = (\delta_{nm})$ is an equicontinuous orthogonal basis for $\ell^p(P)$. If $P \subset \ell^\infty$, then $\ell^p(P)$ contains ℓ^p as a dense $*$ -subalgebra, $\ell^p(P)$ has identity iff $P \subset \ell^p$ iff $\ell^p(P) = \ell^\infty(P)$, and $\ell^\infty(P)$ is a GB^* -algebra. In general, $\ell^p(P)$ is not a GB^* -algebra, though it is symmetric. Two important closed subalgebras of $\ell^\infty(P)$ are $c_0(P) = \{x \in \omega | (x_n a_n) \in c_0 \text{ for all } a \in P\}$ and $c(P) = \{x \in \omega | (x_n a_n) \in c \text{ for all } a \in P\}$.

In [5], a topological algebra A with orthogonal basis (e_n) is called a ϕ -algebra if there exists a family F of sequences with non-negative entries defining a topology on $\phi(A) = \{(e_n^*(x)) | x \in A\}$ by seminorms $p_a((e_n^*(x))) = \sup_n |a_n e_n^*(x)|$, $a \in F$, such that the map $\phi: x \rightarrow \phi(x) = (e_n^*(x)) \in \phi(A)$ is a homeomorphism. It follows from Grothendieck-Pietsch nuclearity criteria that nuclear Köthe algebras $\lambda_1(P)$ and $\lambda_\infty(P)$ supply large classes of ϕ -algebras. As shown in Example 2.8, nuclear $\lambda_\infty(P)$ is a ϕ -algebra without

identity, and $\lambda_1(P)$ is nuclear iff it has identity in which case, it is a ϕ -algebra. It follows from previous paragraph, that for $p \neq 1$ and for P countable, if $\ell^p(P)$ has identity, then it is a ϕ -algebra.

(5.3) Let P satisfy G_1 -condition. For $1 \leq p < \infty$, let $\lambda^p(P) = \{x \in \omega | (x_n a_n^{1/p}) \in \ell^p \text{ for all } a \in P\}$ with topology τ^p defined by seminorms $\{p_{a,p} | a \in P\}$, $p_{a,p}(x) = \|x a^{1/p}\|_p$. Then $(\lambda^p(P), \tau^p)$ too is a topological algebra with $e_n = (\delta_{nm})$ as an orthogonal basis. Similarly the sequence spaces $\lambda^d(P) = \{x \in \omega | \sup_n |x_n a_n|^{1/n} < \infty \text{ for all } a \in P\}$ and $\lambda^\delta(P) = \{x \in \omega | (x_n a_n)^{1/n} \rightarrow 0 \text{ for all } a \in P\}$ are topological algebras with topology defined by seminorms $\{p_{a,d} | a \in P\}$, $p_{a,d}(x) = \sup_n |x_n a_n|^{1/n}$.

(5.4) The space $s = \{x \in \omega | (n^k x_n) \in \ell^\infty \text{ for all } k = 1, 2, \dots\}$ of rapidly decreasing sequences and its cross-dual $s^\times = s' = \{x \in \omega | \sup_n |x_n| n^{-m} < \infty \text{ for some } m\}$ of all tempered sequences are subalgebras of ω with $s \subset \ell^1 \subset \ell^\infty \subset s'$. In fact, with $P = \{(n+1)^k | k \geq 1\}$ a G_∞ -power set, $\lambda(P) = s$ [12, p. 291], and $\lambda(P)^\times = s'$. By [12, Pro. 7.20], for an arbitrary G_1 -power set P , $(\lambda_1(P), \tau_p)$ is nuclear iff $P \subset s$. Thus a nuclear $\lambda_1(P)$ always contains s' as a subalgebra. In particular, s' is a nuclear algebra with an orthogonal basis.

(5.5) For the unit circle T , the Lebesgue space $L^p(T)$, $1 < p < \infty$ is a convolution Banach algebra with orthogonal basis $e_n(t) = t^n$, $n \geq 1$ [8]. Hence the space $L^\omega(T) = \bigcap_{2 \leq p < \infty} L^p(T) = \bigcap_{1 < p < \infty} L^p(T) \neq L^\infty(T)$ is a convolution lmc B_0 -algebra without identity in the topology of L^p -convergence for each $p = 2, 3, 4, \dots$ having orthogonal basis (t^n) . The algebra $L^\omega(T)$ is not nuclear; otherwise, the basis (e_n) , which is already equicontinuous, will be absolute; and Corollary 2.6 (1) would imply that $L^\omega(T)$ has identity.

(5.6) The Frechet space $C^\infty(T)$ of all infinitely differentiable functions on T with the

topology τ defined by the seminorms $p_n(f) = \sup_{t \in T} \left[\sum_{k=0}^n \frac{|f^{(k)}(t)|}{k!} \right]$ is a convolution

*algebra with involution $f \rightarrow f^*$, $f^*(z) = \overline{f(\bar{z})}$. $(C^\infty(T), \tau)$ is a nuclear lmc B_0 -algebra without identity having orthogonal basis (e_n) , $e_n(x) = e^{inx}$, and is homeomorphically isomorphic to the sequence algebra (s, τ_p) . Its dual $D(T)$ consisting of all distributions of T is a convolution *algebra having Dirac delta δ as identity and involution $u \rightarrow u^*$, $\langle u^*, f \rangle = \overline{\langle u, f^* \rangle}$ ($f \in C^\infty(T)$). With the weak topology $\sigma = \sigma(D(T), C^\infty(T))$, $D(T)$ is a locally convex GB^* -algebra with orthogonal basis (e_n) , $e_n(x) = e^{inx}$, and is homeomorphically isomorphic to the sequence algebra s' .

(5.7) Let D be open unit disc. The F -space $H(D)$ of homomorphic functions on D with compact open topology is a nuclear B_0 -algebra with Hadamard product

$$f * g(x) = \frac{1}{2\pi i} \int_{|z|=r} f(z) g(xz^{-1}) z^{-1} dz, x \in D, |x| < r < 1,$$

with identity $e(z) = (1-z)^{-1}$ having orthogonal basis $e_n(z) = z^n$, $n \geq 1$ [8]; and hence is isomorphic to some $(\lambda_1^0(P), \tau_p)$. Its subalgebra $H^\omega(D) = \bigcap_{2 \leq p < \infty} H^p(D) \neq H^\infty(D)$ [17, Ch. 17, Ex. 10] is a Frechet algebra with topology defined by seminorms $f \rightarrow \|f\|_p$,

a Banach algebra with basis (z^n) [8]. The F -space E of entire functions with compact open topology is, with Hadamard product, a nuclear B_0 -algebra having orthogonal basis (z^n) and is isomorphic to $\lambda(P)$ with $P = \{(k^n) | k = 1, 2, \dots\}$ [16]. Similarly the algebra A_0 of complex functions holomorphic at 0 is isomorphic to the cross-dual $\lambda(P)^\times$.

6. Concluding remarks

(6.1) Theorem 2.1 suggests the problem: Is a B_0 -topological algebra A with an orthogonal basis (e_n) functionally continuous in the sense that every multiplicative linear functional on A is continuous (and hence of the form e_n^*)? There exists non-metrizable topological algebras with orthogonal basis admitting discontinuous multiplicative functionals; e.g. $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$. This problem is a variant of Michael problem [14]. Note that if such A is an lmc B_0 -algebra, then A is functionally continuous [22]. If A is a sequentially complete nuclear algebra with 1 and with jointly continuous multiplication, then by Theorem 2.4 and the first paragraph in §2, there is an involution on A making A a GB^* -algebra, so that every multiplicative functional f is positive. Then by [3, §8], f is bounded. Thus if A is a nuclear B_0 -algebra with identity, then A is functionally continuous. Another large class of functionally continuous topological algebras with basis is supplied by the following.

For a countable G_1 -power set P satisfying $0 \neq a_n \leq 1$ for all n , assume $\lambda_1(P)$ to be B_0 and nuclear. Let τ' be the algebraic inductive limit topology on $\lambda_1(P)^\times$, in the sense of [20], defined by the embedding $id_a: (N_a, \|\cdot\|_\infty) \rightarrow \lambda_1(P)^\times$, $\|\cdot\|_\infty$ being the l^∞ -norm on N_a (defined in Example 5.1). Then $(N_a, \|\cdot\|_\infty)$ is a Q -algebra (i.e. quasi-regular elements form an open set). It follows from [20, Prop. 1] that $(\lambda_1(P)^\times, \tau')$ is a functionally continuous lmc algebra. Since $\tau' \leq \tau_P^\times$, $(\lambda_1(P)^\times, \tau_P^\times)$ is functionally continuous.

(6.2) A problem closely related with above is: let (A, τ) be a B_0 -algebra with orthogonal basis (e_n) . Let τ' be a topology on A making (A, τ') a B_0 -algebra. Is $\tau = \tau'$? Note that a topological algebra with orthogonal basis is semi-simple [8]. Hence by standard Banach algebra theory, if both τ and τ' are normable, then $\tau = \tau'$. More generally, if τ is also lmc, then (A, τ) is functionally continuous; and hence by [14, Th. 14.2], $\tau = \tau'$ if τ' is also lmc. Also, assume that each e_n^* is τ' -continuous; in particular, τ' is functionally continuous, or (e_n) is a basis for (A, τ') also. Then via the map $\phi(x) = (e_n^*(x))$ which is one-one, τ and τ' are FK -topologies on (the sequence space) A . It follows from [18, Ch. 3, Th. 2.1], that $\tau = \tau'$.

(6.3) A commutative separable semisimple nuclear B_0 -algebra A need not necessarily admit an orthogonal basis, even if A is lmc and having identity element. Consider the nuclear F -space $H(D)$ of all holomorphic functions on open unit disc D with compact open topology. With pointwise multiplication, $H(D)$ is an lmc B_0 -algebra with identity. It is separable by Example (5.7), and is known to be semisimple. If it admits an orthogonal basis (e_n) , then by [5, Cor. 1.3] its Gelfand space is (e_n^*) , with the result by [14], each $f \in H(D)$ must have a countable spectrum $\sigma(f) = \{e_n^*(f)\}$. On the other hand, $\sigma(f)$ is, in general, uncountable; since it contains range of f . Thus it would be interesting to characterize topological algebras that admit orthogonal basis.

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Critical Sobolev exponent problem in $\mathbb{R}^n (n \geq 4)$ with Neumann boundary condition

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Abstract. In this paper we study the existence and non existence of positive solution for the critical Sobolev exponent problem

$$-\Delta u = u^{(n+2)/(n-2)} + \lambda \alpha(x)u \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 4)$.

Keywords. Critical exponent; flatness condition; Neumann boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. Let $\alpha \in C^\infty(\bar{\Omega})$ and consider the following problem:

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} + \lambda \alpha(x)u \quad \text{in } \Omega \\ u &> 0 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$. Problem (1) with Dirichlet boundary condition instead of Neumann, has been studied by Brezis-Nirenberg [5] in detail. For the Neumann boundary condition, namely, for problem (1), Brezis [3] raised the following question:

“Under what conditions on α and Ω , problem (1) admits a solution?”.

When $n=2$, and the nonlinearity $u^{(n+2)/(n-2)}$ is replaced by $u^2 \exp(bu^2)$, $b > 0$, problem (1) has been studied by authors in [1]. More precisely, in [1] it has been shown that, under suitable assumptions on α , there exists a $\lambda(\alpha) > 0$ such that

$$\begin{aligned} -\Delta u &= u^2 \exp(bu^2) + \lambda \alpha(x)u \quad \text{in } \Omega \\ u &> 0 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \end{aligned} \tag{2}$$

in §4).

Let α change sign in Ω and $\int_{\Omega} \alpha(x) dx < 0$. Let $\lambda(\alpha) > 0$ be the unique real number such that

$$-\Delta \varphi = \lambda(\alpha) \alpha(x) \varphi \quad \text{in } \Omega$$

$$\varphi > 0$$

$$\frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial \Omega$$

admits a solution. For the existence of such $\lambda(\alpha)$, we refer Brown–Lin [6], Senn–Hess [12]. Now by exploiting the techniques used in Brezis–Nirenberg [5], Adimurthi–Yadava [1], we prove the following

Theorem. *Let $n \geq 4$ and assume that*

(i) *α changes sign in Ω and $\int_{\Omega} \alpha(x) dx < 0$. Let $\lambda(\alpha)$ be given by (3).*

(ii) *There exists a $x_0 \in \partial \Omega$ such $\alpha(x_0) > 0$ and $\partial \Omega$ is flat of order $k > 3$ at x_0 .*

Then problem (1) admits a solution $u \in C^2(\bar{\Omega})$ if and only if $\lambda \in (0, \lambda(\alpha))$.

For the meaning of flatness of $\partial \Omega$ of order k at x_0 , see definition (2.1). Here we remark that the flatness condition in the theorem is not satisfied for the ball.

2. Preliminaries

Let $H^1(\Omega)$ denote the usual Sobolev space. For $u \in H^1(\Omega)$ and $1 \leq p \leq 2n/(n-2)$,

$$|\nabla u|_{2,\Omega}^2 = \int_{\Omega} |\nabla u|^2 dx$$

$$|u|_{p,\Omega} = \left(\int_{\Omega} |u|^p dx \right)^{1/p}.$$

To prove the theorem, we need the following

PROPOSITION 2.

Let $\alpha \in C^\infty(\bar{\Omega})$ be such that α changes sign in Ω and $\int_{\Omega} \alpha(x) dx < 0$. Let $\lambda(\alpha)$ be given by (3). Then we have

(i) *for all $\lambda \in (0, \lambda(\alpha))$,*

$$\left\{ \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} \alpha(x) u^2 dx \right\}^{1/2}$$

defines an equivalent norm on $H^1(\Omega)$.

(ii) Let $u \in H^2(\Omega) \cap C(\bar{\Omega})$ be such that

$$\lambda \notin (0, \lambda(\alpha))$$

$$\Delta u + \lambda \alpha u \neq 0$$

$$\Delta u + \lambda \alpha u \leq 0, \quad \frac{\partial u}{\partial \nu} = 0,$$

then u cannot be positive.

For the proof of above proposition we refer Brown–Lin [6] (Theorems 3.10 and 3.11) and Senn–Hess [12] (Proposition 6).

Flatness condition

Let $x_0 \in \partial\Omega$. After a translation and rotation, we assume that $x_0 = 0$ and there exist $R > 0$ and $\rho: B(0, R) \cap \{x_n = 0\} \rightarrow \mathbb{R}$ a smooth function such that

$$\rho(0) = 0, \nabla \rho(0) = 0$$

$$\Omega \cap B(0, R) = \{x \in B(0, R); x_n > \rho(x')\}$$

$$\partial\Omega \cap B(0, R) = \{x \in B(0, R); x_n = \rho(x')\}$$

where $x' = (x_1, \dots, x_{n-1}, 0)$.

DEFINITION 2.1.

We say that $\partial\Omega$ is flat of order k at 0 if $\rho(x') = 0(|x'|^k)$ as $|x'| \rightarrow 0$.

3. Proof of the theorem

Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. Let Γ_0, Γ_1 be disjoint submanifolds of $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and let

$$H^1(\Gamma_0) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}. \quad (4)$$

Let $a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_1)$ be such that

$$\left\{ \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} a u^2 dx + \int_{\Gamma_1} b u^2 dx \right\}^{1/2} \quad (5)$$

defines an equivalent norm on $H^1(\Gamma_0)$. We denote this norm by $\|u\|$. For $u \in H^1(\Gamma_0)$ and $p = (n+2)/(n-2)$, define

$$J(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx \quad (6)$$

$$Q(u) = \frac{\|u\|^2}{\left(\int_{\Omega} |u|^{p+1} dx \right)^{2/(p+1)}} \quad (7)$$

$$S(\Gamma_0, a, b) = \inf \{Q(u); u \in H^1(\Gamma_0) \setminus \{0\}\}. \quad (8)$$

Let

$$S = \inf \left\{ \int_{\Omega} |\nabla u|^2 dx; u \in H_0^1(\Omega), \int_{\Omega} |u|^{2n/(n-2)} dx = 1 \right\} \quad (9)$$

be the best Sobolev constant. Then by Cherrier [7, 8] we have

Lemma 3.1. For every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $u \in H^1(\Omega)$,

$$\left(\int_{\Omega} |u|^{p+1} dx \right)^{2/(p+1)} \leq \left(\frac{2^{2/n}}{S} + \varepsilon \right) \int_{\Omega} |\nabla u|^2 dx + C(\varepsilon) \int_{\Omega} u^2 dx.$$

For the proof of this lemma we also refer Aubin [2] (Theorem 2.30).

Now by using the above mentioned Cherrier's result and Brezis-Lieb lemma [4], we have

Lemma 3.2.

(i) $S(\Gamma_0, a, b) > 0$

(ii) Assume $S(\Gamma_0, a, b) < S/(2^{2/n})$, then there exists a $v \geq 0$ such that $S(\Gamma_0, a, b) = Q(v)$. Further if we define $u_0 = S(\Gamma_0, a, b)^{(n-2)/4} v$, then u_0 satisfies

$$-\Delta u_0 = u_0^{(n+2)/(n-2)} + a(x)u_0 \text{ in } \Omega$$

$$u_0 > 0$$

$$u_0 = 0 \text{ on } \Gamma_0, \quad \frac{\partial u_0}{\partial \nu} + bu_0 = 0 \text{ on } \Gamma_1 \quad (10)$$

and $J(u_0) < (S^{n/2})/2n$.

Proof. (i) By Sobolev imbedding theorem, there exists a constant $C > 0$ such that for all $u \in H^1(\Gamma_0)$,

$$\left(\int_{\Omega} |u|^{p+1} dx \right)^{2/(p+1)} \leq C \|u\|^2.$$

Now (i) follows from the definition of $S(\Gamma_0, a, b)$. (ii) Let $\{u_k\}$ be a minimizing sequence in (8) with $\int_{\Omega} |u_k|^{p+1} dx = 1$. Let for a subsequence, $u_k \rightarrow v$ weakly in $H^1(\Gamma_0)$ and almost everywhere.

Claim 1. $v \neq 0$.

Suppose $v \equiv 0$. Then by Rellich lemma and lemma (3.1) we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx &= \lim_{k \rightarrow \infty} \|u_k\|^2 \\ &= S(\Gamma_0, a, b) \\ &\leq S(\Gamma_0, a, b) \left(\frac{2^{2/n}}{S} + \varepsilon \right) \lim_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx \end{aligned}$$

for every $\varepsilon > 0$. Hence we have,

$$1 \leq S(\Gamma_0, a, b) \left(\frac{2^{2/n}}{S} + \varepsilon \right).$$

Claim 2. $Q(v) = S(\Gamma_0, a, b)$.

Let $v_k = u_k - v$. Then $v_k \rightarrow 0$ weakly and almost everywhere. Now by Rellich lemma

$$\begin{aligned}\|u_k\|^2 &= \|v\|^2 + \|v_k\|^2 + o(1) \\ &= \|v\|^2 + |\nabla v_k|_{2,\Omega}^2 + o(1)\end{aligned}$$

which gives

$$S(\Gamma_0, a, b) = \|v\|^2 + |\nabla v_k|_{2,\Omega}^2 + o(1). \quad (11)$$

By Brezis–Lieb lemma [4] and lemma 3.1, we have

$$\begin{aligned}1 &= |u_k|_{p+1,\Omega}^2 = |v|_{p+1,\Omega}^2 + |v_k|_{p+1,\Omega}^2 + o(1) \\ &\leq |v|_{p+1,\Omega}^2 + \left(\frac{2^{2/n}}{S} + \varepsilon\right) |\nabla v_k|_{2,\Omega}^2 + o(1)\end{aligned}$$

for every $\varepsilon > 0$. Hence

$$\begin{aligned}S(\Gamma_0, a, b) &\leq S(\Gamma_0, a, b) |v|_{p+1,\Omega}^2 + S(\Gamma_0, a, b) \left(\frac{2^{2/n}}{S} + \varepsilon\right) |\nabla v_k|_{2,\Omega}^2 + o(1) \\ &\leq S(\Gamma_0, a, b) |v|_{p+1,\Omega}^2 + |\nabla v_k|_{2,\Omega}^2 + o(1).\end{aligned} \quad (12)$$

Now from (11) and (12), we get

$$\frac{\|v\|^2}{|v|_{p+1,\Omega}^2} \leq S(\Gamma_0, a, b).$$

Hence v is a minimizer in (8).

Since $Q(v) = Q(|v|)$, we may assume $v \geq 0$. Finally if we take $u_0 = S(\Gamma_0, a, b)^{(n-2)/4} v$, then it is easy to check that u_0 satisfies (10) and $J(u_0) < S^{n/2}/2n$. This completes the proof of the lemma.

Lemma 3.3. Let $\alpha \in C(\bar{\Omega})$. Assume that there exists some $x_0 \in \partial\Omega$ such that $\alpha(x_0) > 0$ and $\partial\Omega$ is flat of order $k > 3$ at x_0 . Then for every $\lambda > 0$.

$$S(\lambda\alpha) < \frac{S}{2^{2/n}} \quad (13)$$

where $S(\lambda\alpha) = S(\phi, \lambda\alpha, 0)$.

Proof. Without loss of generality we may assume $x_0 = 0$. Hence $0 \in \partial\Omega$, $\alpha(0) > 0$ and $\partial\Omega$ is flat at 0 of order k which is strictly greater than 3. Let $\rho: B(0, R) \cap \{x: x_n = 0\} \rightarrow \mathbb{R}$ be the function appeared in the definition of flatness.

Let $\varphi \in C_c^\infty(B(0, R/2))$ be such that φ is radial and $\varphi \equiv 1$ on $B(0, R/4)$ and for $\varepsilon > 0$,

$$U_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}},$$

then we claim that there exists a constant $C = C(n) > 0$ such that as $\varepsilon \rightarrow 0$,

$$Q(U_\varepsilon) \leq \begin{cases} \frac{S}{2^{2/n}} - \lambda C \varepsilon + O(\varepsilon^{(k-1)/2}) & \text{if } n \geq 5 \\ \frac{S}{2^{2/n}} - \lambda C \varepsilon |\log \varepsilon| + O(\varepsilon) & \text{if } n = 4 \end{cases} \quad (15)$$

and this implies the lemma.

Proof of (15) follows in several steps. For simplicity we can assume that $\rho \geq 0$. For non-positive ρ 's the estimate (15) follows exactly as in the case of positive ρ . Define

$$\Sigma = \{x \in B(0, R/2); 0 < x_n < \rho(x')\}.$$

Step 1. Let $l \geq 0$, then

$$\int_{\Sigma} \frac{dx}{(\varepsilon + |x|^2)^l} = \begin{cases} |0(1) + O(\varepsilon^{(n+k-1-2l)/2})| & \text{if } n+k-1-2l \neq 0 \\ 0 \left(\log \frac{1}{\varepsilon} \right) & \text{if } n+k-1-2l = 0. \end{cases} \quad (16)$$

Since $\partial\Omega$ is flat of order k at 0, there exists a constant $c > 0$ such that $\rho(x') \leq c|x'|^k$. Therefore we have

$$\begin{aligned} \int_{\Sigma} \frac{dx}{(\varepsilon + |x|^2)^l} &\leq \omega_{n-2} \int_0^R r^{n-2} \left(\int_0^{cr^k} \frac{dx_n}{(\varepsilon + r^2 + x_n^2)^l} \right) dr \\ &= \omega_{n-2} \int_0^R \frac{r^{n-2}}{(\varepsilon + r^2)^{l-1/2}} \left(\int_0^{cr^k/(\varepsilon + r^2)^{1/2}} \frac{1}{(1+t^2)^l} dt \right) dr \\ &\leq c \omega_{n-2} \int_0^R \frac{r^{n-2+k}}{(\varepsilon + r^2)^l} dr \\ &= c \omega_{n-2} \varepsilon^{(n+k-1-2l)/2} \int_0^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l} dr. \end{aligned} \quad (17)$$

Now

$$\begin{aligned} \int_0^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l} dr &= \int_0^1 \frac{r^{n-2+k}}{(1+r^2)^l} dr + \int_1^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l} dr \\ &= 0(1) + 0 \left(\int_1^{R/\varepsilon^{1/2}} r^{n-2+k-2l} dr \right) \\ &= 0(1) + \begin{cases} 0(1) + O(\varepsilon^{(-n-k+1+2l)/2}) & \text{if } n+k-1-2l \neq 0 \\ 0 \left(\log \frac{1}{\varepsilon} \right) & \text{if } n+k-1-2l = 0 \end{cases} \end{aligned} \quad (18)$$

Hence from (18) and (17) we have (16).

Step 2. Since $k > 2$, without loss of generality we can assume that $k = 3 + \delta$, $0 < \delta < 1$. Then

$$\int_{\Sigma} |U_{\varepsilon}|^2 dx = 0(1) + 0(\varepsilon^{(k+3-n)/2}) \quad (19)$$

$$\int_{\Sigma} |\nabla U_{\varepsilon}|^2 dx = 0(1) + 0(\varepsilon^{(k+1-n)/2}) \quad (20)$$

$$\int_{\Sigma} |U_{\varepsilon}|^{2n/(n-2)} dx = 0(1) + 0(\varepsilon^{(k-1-n)/2}). \quad (21)$$

Since

$$\nabla U_{\varepsilon} = \frac{\nabla \varphi}{(\varepsilon + |x|^2)^{(n-2)/2}} - \frac{(n-2)x}{(\varepsilon + |x|^2)^{n/2}},$$

hence we have

$$|U_{\varepsilon}|^2 = 0\left(\frac{1}{(\varepsilon + |x|^2)^{n-2}}\right), \quad |\nabla U_{\varepsilon}|^2 = 0\left(\frac{1}{(\varepsilon + |x|^2)^{n-1}}\right)$$

and

$$|U_{\varepsilon}|^{2n/n-2} = 0\left(\frac{1}{(\varepsilon + |x|^2)^n}\right).$$

Therefore from step (1), by taking $l = n - 2$, $n - 1$ and n , we obtain (19), (20) and (21) respectively.

Step 3. From Brezis–Nirenberg [5] (see page 144, eqs (1.11), (1.12) and (1.13)), there exist positive constants k_1, k_2 and k_3 such that $k_1/k_2 = S$ and

$$|\nabla U_{\varepsilon}|_{2, B(0, R)}^2 = \frac{k_1}{\varepsilon^{(n-2)/2}} + 0(1) \quad (22)$$

$$|U_{\varepsilon}|_{2n/(n-2), B(0, R)}^2 = \frac{K_2}{\varepsilon^{(n-2)/2}} + 0(1) \quad (23)$$

$$|U_{\varepsilon}|_{2, B(0, R)}^2 = \begin{cases} \frac{k_3}{\varepsilon^{(n-4)/2}} + 0(1) & \text{if } n \geq 5 \\ k_3 |\log \varepsilon| + 0(1) & \text{if } n = 4. \end{cases} \quad (24)$$

Now from (19) to (24) we have

$$\begin{aligned} |\nabla U_{\varepsilon}|_{2, \Omega}^2 &= \frac{1}{2} |\nabla U_{\varepsilon}|_{2, B(0, R)}^2 - |\nabla U_{\varepsilon}|_{2, \Sigma}^2 \\ &= \frac{k_1}{2\varepsilon^{(n-2)/2}} [1 + 0(\varepsilon^{(n-2)/2}) + 0(\varepsilon^{(k-1)/2})] \end{aligned} \quad (25)$$

$$= \begin{cases} \frac{k_3}{2e^{(n-4)/2}} + [1 + 0(e^{(k-1)/2}) + 0(e^{(n-4)/2})] & \text{if } n \geq 5 \\ \frac{k_3}{2} |\log \varepsilon| + 0(1) + 0(e^{(k+3-n)/2}) & \text{if } n = 4 \end{cases}$$

$$\begin{aligned} |U_\varepsilon|_{2n/(n-2), \Omega}^{2n/(n-2)} &= \frac{1}{2} |U_\varepsilon|_{2n/(n-2), B(0, R)}^{2n/(n-2)} - |U_\varepsilon|_{2n/(n-2), \Sigma}^{2n/(n-2)} \\ &= \frac{1}{2} \left(\frac{k_2}{\varepsilon^{(n-2)/2}} \right)^{n/(n-2)} [1 + 0(\varepsilon^{n/2}) + 0(\varepsilon^{(k-1)/2})]. \end{aligned}$$

Hence

$$|U_\varepsilon|_{2n/(n-2), \Omega}^2 = \frac{k_2}{2^{(n-2)/n} \varepsilon^{(n-2)/2}} [1 + 0(\varepsilon^{n/2}) + 0(\varepsilon^{(k-1)/2})].$$

Now choose $R > 0$ and $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for all x in $B(0, R) \cap \bar{\Omega}$, then

$$Q(U_\varepsilon) \leq \frac{|\nabla U_\varepsilon|_{2, \Omega}^2 - \lambda \alpha_0 |U_\varepsilon|_{2, \Omega}^2}{|U_\varepsilon|_{2n/(n-2), \Omega}^2}.$$

From (25) to (28) we have

$$Q(U_\varepsilon) \leq \begin{cases} \frac{S}{2^{2/n}} - \frac{\lambda \alpha_0 k_3}{2^{2/n} k_2} \varepsilon + 0(\varepsilon^{(k-1)/2}) & \text{if } n \geq 5 \\ \frac{S}{2^{2/n}} - \frac{\lambda \alpha_0 k_3}{2^{2/n} k_2} \varepsilon |\log \varepsilon| + 0(\varepsilon) & \text{if } n = 4. \end{cases}$$

Let $C = \lambda \alpha_0 k_3 / 2^{2/n} k_2$, then (15) follows from (29). This proves the claim and hence the lemma.

Proof of the Theorem. Let $\lambda \in (0, \lambda(\alpha))$. By (i) of proposition 2,

$$\left\{ \int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} \alpha(x) u^2 dx \right\}^{1/2}$$

defines an equivalent norm on $H^1(\Omega)$. From lemma 3.3, $S(\lambda\alpha) < (S/2^{2/n})$. Hence from (ii) of lemma 3.2, there exists a $u_0 \in H^1(\Omega)$, $u_0 \geq 0$ which solves (1). By Cherrier [1], $u_0 \in C^\infty(\bar{\Omega})$ and by maximum principle $u_0 > 0$ in $\bar{\Omega}$. On the other hand, if $\lambda \notin (0, \lambda(\alpha))$, then by (ii) of proposition 2, (1) does not admit any solution. This completes the proof of the theorem.

4. Concluding remarks

1. Similar construction as in Brezis [3] (See page 21 example (2)), it is possible to construct a $\alpha(x) < 0$ such that (1) admits a solution for $\lambda = 1$. We do not know how to deal with (1) when $\lambda\alpha(x) \leq 0$.

2. By a result of Senn-Hess [2] (See page 462, proposition (6)), it follows that if $\int_{\Omega} \alpha(x) dx = 0$, then for any $\lambda \in \mathbb{R}$, (1) does not admit a solution.

3. Following the method of this paper and using a recent result of Escobar [10], it is possible to show that under suitable flatness assumptions at a boundary point,

$$-\Delta u = \lambda \alpha(x) u \text{ in } \Omega$$

$$u > 0$$

$$\frac{\partial u}{\partial \nu} = u^{n/n-2} \text{ on } \partial\Omega$$

admits a solution for all $\lambda \in (0, \lambda(\alpha))$.

4. Let $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \phi$, Γ_0 and Γ_1 the smooth submanifolds of dimension $n-1$. Consider the following mixed boundary value problem

$$-\Delta u = u^{(n+2)/(n-2)} + \alpha(x)u + \mu u \text{ in } \Omega$$

$$u > 0$$

$$u = 0 \text{ on } \Gamma_0$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1. \quad (30)$$

Let x_0 be in the interior of Γ_1 and assume that $\partial\Omega$ is flat at x_0 of order k strictly greater than 3 and $\alpha(x_0) \geq 0$. Further assume that $-\Delta - \alpha$ is positive on $H^1(\Gamma_0)$. Let μ_1 be the first eigenvalue of $-\Delta - \alpha$ on $H^1(\Gamma_0)$. Then by similar method used in this paper it follows that for $n \geq 4$, $\mu \in (0, \mu_1)$, (30) admits a weak solution.

However, it should be noted that under a stronger assumption on α , viz $\alpha(x_0) > 0$, (30) admits a solution even for $\mu = 0$.

When $\alpha \equiv 0$, $\mu = 0$, Lions-Pacella-Tricarico [11] have proved that, under suitable assumption on Γ_0 and Γ_1 , with $\Gamma_0 \neq \phi$, problem (30) admits a solution.

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Anderson model with decaying randomness existence of extended states

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Abstract. We study the Anderson model with decaying randomness in $v \geq 3$ dimensions and show that there is absolutely continuous spectrum in $[-2v, 2v]$. The distribution of the potentials is assumed to have finite variance and the coupling constants decay at infinity at a rate $\alpha > 1$.

Keywords. Anderson model; decaying randomness; extended states.

1. Introduction

In the last two decades scattering theory of finite particle quantum systems has been well developed and as an offshoot several techniques were developed for the study of spectral properties of self-adjoint operators. Trace class theory is one such tool that was effectively used in a beautiful way by Simon and Spencer [18] recently to show the absence of absolute continuous spectrum for unbounded potentials in one-dimensional discrete systems.

The spectral properties of the Anderson-like systems have proved elusive in higher dimensions in the low disorder regime. While the expected results are that for small disorder there is some absolute continuous spectrum near zero energy and pure point spectrum elsewhere, only the pure point nature of the spectrum was proved in the last few years, after a powerful estimate on the Green function was obtained by Fröhlich and Spencer [9], Simon and Wolff [19], Fröhlich *et al* [8], Delyon *et al* [6] and Carmona *et al* [3]. The only higher dimensional result is from Kunz and Souillard [10] who exhibited Anderson – Mott transition on the Bethe lattice, which in some sense represents the ∞ -dimensional case.

In the one-dimensional model rich theory exists. There are excellent reviews of these results in Cycon *et al* [4], Carmona [2] and Spencer [20] and we refer to them for a complete set of references for the one-dimensional theory. The consequences arising from the presence of absolute continuous spectrum was investigated by Kotani [11] in continuous models and Simon [16] for the discrete case. On the other hand Simon [17] and Delyon *et al* [7] studies the spectral properties for various decaying random potentials in one-dimension. They obtain various spectral types along with information on the decay rate of eigenfunctions. Similar studies were done by Kotani and Ushiroya [12] for the continuous models. In addition to these results Deift and Simon [5] study the absolutely continuous spectrum for some almost periodic potentials and Claude–Allen Pillet [13] studied the absolute continuous spectrum

In this paper we study the time evolution of regions in the configuration space $l^2(\mathbb{Z}^v)$, $v \geq 3$ and show the presence of absolute continuous spectrum in the Anderson-like models with the *coupling constants decaying at infinity at a rate bigger than 1*. Our strategy in this paper is to exploit the fact, well known in scattering theory of finite particle quantum systems, that existence of wave operators $\Omega(A, B)$ for two self-adjoint operators A and B will show that $\sigma_{ac}(B) \subset \sigma_{ac}(A)$, by the intertwining property of the wave operators. Our paper consists mainly of two sections, in both of them the coupling constants are assumed to decay at infinity at a rate $\alpha > 1$. In the first we present a proof for potentials with bounded support where we do not really assume anything about the randomness in the potentials (other than that the support of their distribution is bounded). In the next section we consider independent random variables and allow their distribution to have unbounded support but finite variance.

Throughout this paper we follow the convention that if $n \in \mathbb{Z}^v$, then $|n|$ denotes the norm in $\mathbb{Z}^v$ namely, $|n| = (n_1^2 + \dots + n_v^2)^{1/2}$. For a subset S of $\mathbb{R}^v$ we denote by $F(S)$ the orthogonal projection in $l^2(\mathbb{Z}^v)$ corresponding to $S \cap \mathbb{Z}^v$.

2. Bounded potentials

We consider the operator $H^\omega = \Delta + V^\omega$ on $l^2(\mathbb{Z}^v)$, $v \geq 3$, where we define the discrete Laplacian Δ by

$$(\Delta u)(n) = \sum_{|i-n|=1} u(i). \quad (1)$$

Then the operator Δ is bounded self-adjoint and has purely absolutely continuous spectrum $[-2v, 2v]$. We consider the potentials V^ω as multiplication operators and the corresponding functions are required to satisfy the following condition, where $q^\omega(n)$ is bounded measurable real valued function, on some probability space, $(\Omega, \mathfrak{B}, d\mu)$ for each $n \in \mathbb{Z}^v$.

$$V^\omega(n) = a_n q^\omega(n), \quad \text{with } |a_n| \leq \frac{\text{const.}}{(1 + |n|)^\alpha}, \alpha > 1 \text{ as } |n| \rightarrow \infty. \quad (2)$$

Under the above conditions on the potentials, H^ω will be a bounded self-adjoint operator on $l^2(\mathbb{Z}^v)$. For the following theorem we note that by Stone's theorem the one parameter group $\exp(iAs)$, generated by a self-adjoint operator A , is strongly differentiable in s .

Theorem 2.1. *Let $v \geq 3$ and let $V^\omega(n)$ be of the form given in (2). Then the wave operators*

$$\Omega_+ = \lim_{t \rightarrow +\infty} \exp(iH^\omega t) \exp(-i\Delta t) \text{ exist } \forall \omega \in \Omega.$$

Proof. To prove the theorem it suffices to show that $\exp(iH^\omega t) \exp(-i\Delta t)f$ is strongly *cauchy* for any $f \in l^2(\mathbb{Z}^v)$. Since $\{\delta_n, n \in \mathbb{Z}^v\}$ forms an orthonormal basis for $l^2(\mathbb{Z}^v)$, finite

linear combinations of these vectors are dense in $l^2(\mathbb{Z}^v)$, therefore by density arguments without loss of generality we may take $f = \delta_k$. We will further take $f = \delta_0$, since we are going to be interested in large time behaviour in the following, it will be clear that no generality will be lost by doing so. Then by the fundamental theorem of calculus,

$$\begin{aligned} & \|\exp(iH^\omega t)\exp(-i\Delta t) - \exp(iH^\omega u)\exp(-i\Delta u)\delta_0\| \\ &= \left\| \int_u^t ds \exp(iH^\omega s)(iH^\omega - i\Delta)\exp(-i\Delta s)\delta_0 \right\| \\ &\leq \int_u^t ds \left\| \exp(iH^\omega s) V^\omega \exp(-i\Delta s)\delta_0 \right\|. \end{aligned} \quad (3)$$

Now,

$$\begin{aligned} \|\exp(iH^\omega s) V^\omega \exp(-i\Delta s)\delta_0\| &\leq \|V^\omega F(|n| \geq \beta s)\exp(-i\Delta s)\delta_0\| \\ &\quad + \|V^\omega F(|n| \leq \beta s)\exp(-i\Delta s)\delta_0\| \end{aligned}$$

for some $\beta > 0$ sufficiently small compared to 1. Our conditions (2) on the potentials immediately show that,

$$\|V^\omega F(|n| \geq \beta s)\exp(-i\Delta s)\delta_0\| \leq \frac{C}{(1+s)^\alpha} \text{ independent of } \omega \quad (4)$$

as long as β is strictly positive. Now,

$$\begin{aligned} & \|V^\omega F(|n| \leq \beta s)\exp(-i\Delta s)\delta_0\|^2 \\ &= \sum_{|n| \leq \beta s} V^\omega(n)^2 \overline{\langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle} \langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle. \end{aligned} \quad (5)$$

Clearly by Proposition A.4

$$\langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle = \prod_{i=1}^v (-i)^{n_i} (J_{n_i}(2s)) = \prod_{i=1}^v (i)^{n_i} (J_{-n_i}(2s)),$$

$n = (n_1, \dots, n_v)$, which gives us the estimate

$$\begin{aligned} & \|V^\omega F(|n| \leq \beta s)\exp(-i\Delta s)\delta_0\|^2 \\ &\leq \sum_{i=1}^v C_1 \sum_{0 < n_i \leq \beta s} \frac{C}{(1+|n|)^{2\alpha}} \prod_{i=1}^v |J_{n_i}(s)|^2, \quad \alpha > 1. \end{aligned}$$

Now using Proposition A.3 we see that

$$\|V^\omega F(|n| \leq \beta s)\exp(-i\Delta s)\delta_0\|^2 \leq \frac{C}{(1+s)^{2+\varepsilon}} \{1 + O(1)\} \quad \text{for some } \varepsilon > 0.$$

Once we have this estimate, using inequalities (3), (4) and (5) we can immediately conclude that

$$\lim_{u, t \rightarrow \infty} \|(\exp(itH^\omega)\exp(-i\Delta t) - \exp(iuH^\omega)\exp(-i\Delta u))\delta_0\| \rightarrow 0.$$

#

Theorem 2.2. Let $H^\omega = \Delta + V^\omega$, with V^ω satisfying condition (2). Then, $\sigma_{ac}(H^\omega) = [-2v, 2v]$ for each ω .

Proof. Since the wave operator exists for the (H^ω, Δ) pair by Theorem 2.1, applying Theorem A.1 shows that

$$\sigma_{ac}(H^\omega) \supset \sigma_{ac}(\Delta) = [-2v, 2v].$$

On the other hand since $\forall n$, $V^\omega(n)$ is bounded, and goes to zero as $|n| \rightarrow \infty$, V^ω is a compact operator on $l^2(\mathbb{Z}^v)$. Hence H^ω has the same essential spectrum as Δ by Weyl criterion (see Reed and Simon [14]). Therefore $[-2v, 2v] = \sigma_{ess}(H^\omega)$. But $\sigma_{ess}(H^\omega) \supset \sigma_{ac}(H^\omega)$, hence proving the theorem. $\#$

3. Potentials with finite variance

In this section we consider potentials V^ω acting as multiplication by the functions $V^\omega(n) = a_n q^\omega(n)$, which are independent random variables satisfying the following conditions, $\{a_n\}_{n \in \mathbb{Z}^v}$ is a real valued sequence such that

$$|a_n| \leq \frac{C}{(1 + |n|)^\alpha}, \quad \alpha > 1 \quad (6)$$

and $\{q^\omega(n)\}_{n \in \mathbb{Z}^v}$ are i.i.d. random variables, distributed according to a probability measure $d\mu$ satisfying

$$\int x d\mu(x) = 0 \quad \text{and} \quad \int x^2 d\mu(x) = \sigma^2 < +\infty. \quad (7a)$$

For a special case, we also consider $d\mu$ satisfying

$$\int_{\mathbb{R}} |x|^p d\mu < +\infty \quad \text{for } p > 2. \quad (7b)$$

In the following we consider potentials V^ω given as multiplication by $V^\omega(n)$ on $l^2(\mathbb{Z}^v)$. Then the operator $H^\omega = \Delta + V^\omega$ is self-adjoint on $l^2(\mathbb{Z}^v)$ with the set of finite linear combinations of $\{\delta_k\}$, $k \in \mathbb{Z}^v$ forming a core for the operator.

Remark 3.1. At this stage let us say that the conditions (7a) would suffice for existence of absolutely continuous spectrum. We take the later condition to say more about the spectrum when the distribution has higher order moments (as in the case of Gaussian for example). We also note here that $d\mu$ need not have a density with respect to the Lebesgue measure and is allowed to be sum of point measures in which case the integral in the definition of the mean etc., (in conditions (7a) and (7b) is replaced by a sum.

Before we present the main theorem of this section we state and prove the following technical lemma.

$$V^\omega \exp(-i\Delta t)\delta_k \in l^2(\mathbb{Z}^\nu) \text{ a.e. } \omega$$

whenever V^ω satisfies condition (7a)

Proof. Let us recall that $(V^\omega \delta_m) = V^\omega(m) \delta_m \forall m \in \mathbb{Z}^\nu$ and $V^\omega(m) = a_m q^\omega(m)$, $\{q^\omega(m)\}_{m \in \mathbb{Z}^\nu}$ i.i.d. random variables. So consider the events

$$\Delta_m = \{\omega : |q^\omega(m)| > (1 + |m|)^{2(\nu + \varepsilon)}\}, m \in \mathbb{Z}^\nu.$$

Then we have by Cauchy–Schwarz inequality,

$$\text{Prob}(\Delta_m) = \int_{\Delta_m} \frac{1}{x} \cdot x d\mu(x) \leq \frac{\sigma}{(1 + |m|)^{\nu + \varepsilon}}, \forall m \in \mathbb{Z}^\nu.$$

Hence,

$$\sum_{m \in \mathbb{Z}^\nu} \text{Prob}(\Delta_m) \leq \sum_{m \in \mathbb{Z}^\nu} \frac{\sigma^2}{(1 + |m|)^{\nu + \varepsilon}} < +\infty.$$

Therefore by Borel–Cantelli lemma only finitely many events Δ_m can occur. Therefore $\Omega_0 = \Omega \setminus (\cup \Delta_m)$ has full measure. Now, consider the set $S \subset l^2(\mathbb{Z}^\nu)$ of finite linear combination of $\delta_n, n \in \mathbb{Z}^\nu$. S is a dense linear space in $l^2(\mathbb{Z}^\nu)$ and on this set V^ω is well defined, so for $f \in S$, let us look at

$$\sup_{\{f \in S, \|f\| = 1\}} |\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 \quad \omega \in \Omega_0.$$

If this is finite then the lemma follows. To show this let us take $f = \sum \alpha_i \delta_i$, where the sum ranges over a finite set in $\mathbb{Z}^\nu$. Then we have,

$$|\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 \leq \left| \sum \alpha_i \langle V^\omega \delta_i, \exp(-it\Delta)\delta_k \rangle \right|^2$$

By Cauchy–Schwarz inequality we have,

$$\begin{aligned} |\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 &\leq \left(\sum \alpha_i^2 \right) \left(\sum_{i \in \mathbb{Z}^\nu} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \\ &= \|f\|^2 \left[\left(\sum_{i \in S^\omega} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \right. \\ &\quad \left. + \left(\sum_{i \notin S^\omega} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \right]. \end{aligned} \quad (8)$$

Since S^ω is a finite set in $\mathbb{Z}^\nu$, the first term in the above inequality is bounded by some finite C_ω for each $\omega \in \Omega_0$. The second term is bounded by

$$\sum_{i \in \mathbb{Z}^\nu \setminus S^\omega} (1 + |i|)^{\nu + \varepsilon} |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2,$$

which is finite for each fixed t and k , since there is exponential decay in i for $|\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|$, as $|i| \rightarrow \infty$, by proposition A.2. #

Lemma 3.3. *Consider a $k \in \mathbb{Z}^v$ fixed. Then, we have*

$$\int \|\mathcal{V}^\omega \exp(-i\Delta t)\delta_k\| dt < \infty \text{ a.e. } \omega,$$

whenever $\mathcal{V}^\omega$ satisfies condition (7a).

Proof. Let us consider the random variable,

$$\|\mathcal{V}^\omega \exp(-i\Delta t)\delta_k\|^2$$

which is well defined a.e. ω by Lemma 3.2 and note that it suffices to by Fubini and Cauchy-Schwarz inequality that

$$\mathbb{E} \left\{ \int \|\mathcal{V}^\omega \exp(-i\Delta t)\delta_k\|^2 dt \right\} \leq \int dt \mathbb{E} \{ \|\mathcal{V}^\omega \exp(-i\Delta t)\delta_k\|^2 \} < \infty,$$

to get the almost everywhere statement mentioned above. Now by Fubini we have,

$$\mathbb{E} \{ \|\mathcal{V}^\omega \exp(-i\Delta t)\delta_k\|^2 \} = \|V_\sigma \exp(-i\Delta t)\delta_k\|^2,$$

where V_σ is bounded operator on $l^2(\mathbb{Z}^v)$ given by,

$$(V_\sigma u)(m) = a_m \{ \mathbb{E}(q^\omega(m))^2 \}^{1/2} u(m),$$

and a_m is as in condition (6). Now the required estimate is obvious by proceeding as in the proof of Theorem (2.1). #

We now state the main theorem of this section, namely.

Theorem 3.4. *Consider the operator $H^\omega = \Delta + \mathcal{V}^\omega$ on $l^2(\mathbb{Z}^v)$, $v \geq 3$. Then the following are valid.*

- (i) *If $\mathcal{V}^\omega$ satisfies condition (7a), then $[-2v, 2v] \subseteq \sigma_{ac}(H^\omega)$.*
- (ii) *If $\mathcal{V}^\omega$ satisfies condition (7b) in addition to (7a) with $pa > v$, then*

$$\sigma_{ac}(H^\omega) = [-2v, 2v]$$

Proof.

- (i) The proof of this theorem follows from Proposition A.1, if we show that the strong limits,

$$\lim_{t \rightarrow \infty} \exp(itH^\omega) \exp(-it\Delta) \text{ exist.}$$

To show this we need only show that, (as in Theorem 2.1)

$$\lim_{u, t \rightarrow \infty} \|(\exp(itH^\omega) \exp(-it\Delta) - \exp(iuH^\omega) \exp(-iu\Delta))\delta_0\| \rightarrow 0.$$

Now by the fundamental theorem of calculus and Lemma 3.2 we have the inequality,

$$\begin{aligned} & \|(\exp(itH^\omega)\exp(-it\Delta) - \exp(iuH^\omega)\exp(-iu\Delta))\delta_0\| \\ & \leq \int_u^t ds \|V^\omega \exp(-is\Delta)\delta_0\| \text{ a.e. } \omega \end{aligned}$$

from which using Lemma 3.3 the result follows.

(ii) If $\alpha p > \nu$, then we already know by (i) $\sigma_{ac}(H^\omega) \supset [-2\nu, 2\nu]$. Now consider the sequence $|V^\omega(n)|^p, n \in \mathbb{Z}^+$. This sequence goes to zero as $|n| \rightarrow \infty$ almost everywhere, since

$$\mathbb{E} \left\{ \sum |V^\omega(n)|^p \right\} \leq \sum \mathbb{E} \{ |V^\omega(n)|^p \} \leq \sum \frac{\mathbb{E} \{ |q^\omega(n)|^p \}}{(1+|n|)^{\alpha p}} < \infty$$

by conditions (7b). Hence the sequence $V^\omega(n)$ also goes to zero almost everywhere as $|n| \rightarrow \infty$, showing that V^ω is a compact operator almost every ω . The rest of the result then follows from Weyl's theorem. $\#$

Remark 3.5. In case (i) of the above theorem we cannot say what happens outside $[-2\nu, 2\nu]$ at the moment. We note here that according to Simon's analysis in [17] where he explains Pearson's intuition, the quantity Σa_n^2 being finite or not plays a role in the presence of continuous spectrum in one dimension. We note here that we actually have $\Sigma a_n^2 = \infty$. We also note that the condition (7a) does not mean that $V^\omega(n) \sim O(a_n)$ a.e. as $|n| \rightarrow \infty$.

4. Conclusion

In this paper we have presented a time dependent method for studying the absolutely continuous spectrum of some Anderson models in higher dimensions. The proofs show that it is sufficient to study the free evolution of the random potential in bounded, though increasing in time, regions in the configuration space. Our results show that the time dependent study of the disordered system is a powerful tool, for handling the absolutely continuous spectrum. It is also quite clear that these techniques require the dimension to be bigger than 2, otherwise the free evolution does not have integrable decay in time, a fact crucial for the method to work. To this extent it seems that the role of higher dimension in producing absolutely continuous spectrum in the Anderson model is clear. The small energy condition is also clear from the method. Though we run the risk of being speculative, we would like to mention that heuristically, the sum

$$\sum_{|n| \leq \beta s} V^\omega(n) \langle \delta_n, \exp(-is\Delta)\delta_0 \rangle$$

converges in distribution, as $s \rightarrow \infty$, being the sum of independent mean zero random variables. When the disorder goes to zero, most of the terms in the sum are very small, since the potential is small with large probability in that case, so it might be possible to do a scaling analysis similar to the one done by Fröhlich–Spencer to control the growth of $\Sigma V^\omega(n)^2 (1+s)^{-\nu+2} F(|n| \leq \beta s)$, in s , as the disorder goes to zero. This would be sufficient to show the existence of absolutely continuous spectrum in the case of homogeneous potentials.

Though we still cannot handle the case of homogeneous potentials, i.e. no decay in the coupling constants, it is still worth investigating if these methods can yield absolutely continuous spectrum in that case also as also the case when the coupling constants decay slower than the rate we considered here. Since the intuition for the techniques employed here is borrowed from scattering theory, it is reasonable to conjecture that: *If the coupling constants decay at infinity at a rate $\alpha > 0$ then $\sigma_{ac}(H^\omega) \supset [-2v, 2v]$ when $v \geq 3$.*

Acknowledgments

The author thanks Professors B Simon, S Kotani, R Carmona for discussions, at various times in the past, which contributed to my understanding of the disordered systems. Prof. Pl. Muthuramalingam for many useful discussions and Prof. Kotani for correcting Lemma 2.3 in a previous version of the paper.

Appendix

PROPOSITION A.1.

If A and B are two self-adjoint operators on a Hilbert space with B bounded and suppose the limits $\lim_{t \rightarrow \infty} \exp(iAs) \exp(-iBs) E_{ac}(B)$ exist. Then $\sigma_{ac}(B) \subseteq \sigma_{ac}(A)$.

In the above proposition we take B bounded since it is enough for our purposes. This is a standard result in scattering theory and the proof involves the fact that the limits mentioned above intertwine the operators A and B . The reader is referred to (Proposition I, XI-3) Reed and Simon [14] or (Proposition 5.21) Amrein *et al* [1] for a proof.

PROPOSITION A.2.

Suppose A is a bounded self-adjoint operator on $l^2(\mathbb{Z}^v)$ such that $\langle \delta_{k'}, A \delta_k \rangle = 0$ whenever $|k - k'| > 1$ for a fixed 1, then for any fixed $n' \in \mathbb{Z}^v$, there exist $\alpha_1 > 0$ and $C_2 > 1$ such that

$$|\langle \delta_n, \exp(iAs) \delta_{n'} \rangle| \leq C_1(n') \exp(-\alpha_1 |n|), \quad \text{as } |n| \rightarrow \infty$$

whenever $n \in \mathbb{Z}^v$ satisfies $|n - n'| > C_2(\|A\|s \cdot l + 1)$.

Proof. Notice that since $|\langle \delta_k, A \delta_{k'} \rangle| = 0$ if $|k - k'| > 1$,

$$\langle \delta_n, A^m \delta_{n'} \rangle \equiv 0 \text{ whenever } |n - n'| > m \cdot 1.$$

Hence, when $[|n - n'|] = M \cdot l + 1$, we have

$$\langle \delta_n, \exp(iAs) \delta_{n'} \rangle = \left\langle \delta_n, \left\{ \exp(iAs) - \sum_{k=0}^M \frac{(iAs)^k}{k!} \right\} \delta_{n'} \right\rangle.$$

Using the spectral theorem for bounded self-adjoint operators and the inequality

$$|\exp(i\lambda s) - \sum_{k=0}^M \frac{(i\lambda s)^k}{k!}| \leq C_1 \frac{|\lambda s|^M}{M!}$$

$$M! \simeq \sqrt{2\pi M} \cdot M^M \exp(-M), \text{ as } M \rightarrow \infty$$

for the factorial of a large integer we see that

$$\begin{aligned} & |\langle \delta_n, \exp(iAs) \delta_{n'} \rangle| \\ & \leq C_1 \sup \{ M^{-1/2} \exp(M \{ \ln |s| \}) \exp(-M \ln M) \exp(M) \} \\ & = C_1 \exp(M \ln) \left(\left\{ \frac{\|A\| |s|}{M} \right\} \right) M^{-1/2}. \end{aligned}$$

Now taking $|n - n'| > \exp \cdot \|A\| s \cdot 1 + 2$, we see that $\ln(\|A\| |s| \cdot l \cdot \exp) / ([|n - n'|] - 1) < 0$. Since l is a fixed finite number, we get,

$$|\langle \delta_n, \exp(iAs) \delta_0 \rangle| \leq C_1 \exp(-\alpha_1 |n|)$$

as claimed in the proposition. #

In the following $J_k(r)$ will denote the Bessel function of integral order k and argument r .

PROPOSITION A.3.

Let $\alpha > 1$, $\beta_1 \ll 1$ be given and consider s sufficiently large. Then $\exists \varepsilon > 0$ such that for $n \in \mathbb{Z}^v$,

$$\sum_{|n| \leq \beta_1 s} \prod_{i=1}^v |J_{n_i}(s)|^2 \frac{1}{(1 + |n|)^{2\alpha}} \leq \frac{c}{(1 + s)^{2+\varepsilon}} \{1 + o(1)\}$$

Proof. Note that $J_{-k}(s) = (-1)^k J_k(s)$ for $k \in \mathbb{Z}^+$. Hence it suffices to consider $n_i \leq \beta_1 s$ $\forall i = 1, \dots, v$. Now the asymptotic expansions for Bessel function (cf. Watson [21] Section 8.12) show that for $k/s \ll 1$,

$$|J_k(s)| \leq \frac{C}{\sqrt{(2\pi s)}} \{1 + o(1)\}.$$

Therefore a crude estimate shows that for any $\beta_1 \ll 1$ and $n \in \mathbb{Z}^v$,

$$\begin{aligned} \sum_{|n| \leq \beta_1 s} \prod_{i=1}^v |J_{n_i}(s)|^2 \frac{1}{(1 + |n|)^{2\alpha}} & \leq \frac{C(1 + o(1))}{(1 + s)^v} \sum_{|n| \leq \beta_1 s} \frac{1}{(1 + |n|)^{2\alpha}} \\ & \leq \frac{C_1(1 + o(1))}{(1 + s)^v} \cdot \sum_{i=1}^v \sum_{\substack{n_i \in \mathbb{Z} \\ 0 < n_i \leq \beta_1 s}} \frac{1}{(1 + n_i)^{2\alpha}} \\ & \leq \frac{c_2(1 + o(1))}{(1 + s)^{2+\varepsilon}}. \end{aligned}$$

since $\alpha > 1$, the last step follows for a suitable $\varepsilon > 0$.

PROPOSITION A.4.

Let $n, m \in \mathbb{Z}^v$ and Δ as in (1), then we have

$$\langle \delta_n, \exp(-i\Delta s) \delta_m \rangle = \prod_{i=1}^v (-i)^{(n-m)_i} J_{(n-m)_i}(2s)$$

where $J_k(s)$ is the Bessel function of integral order k .

Proof. First note that the Fourier transform U gives the equality

$$(U \Delta U^{-1} u)(k) = \left(2 \sum_{i=1}^v \cos k_i \right) u(k).$$

Hence a simple computation shows that,

$$\langle \delta_n, \exp(-i\Delta s) \delta_m \rangle = \frac{1}{(2\pi)^v} \int dk \exp(i(n-m) \cdot k) \exp\left(-i2s \sum_{i=1}^v \cos k_i\right).$$

Since $dk = \prod_{i=1}^v dk_i$, $k_i \in [-\pi, \pi]$, we have

$$\langle \delta_n, \exp(-i\Delta s) \delta_m \rangle = \prod_{i=1}^v \left\{ \frac{1}{(2\pi)} \int_{-\pi}^{\pi} dk_i \exp(i(n-m)_i k_i) \exp(-i2s \cos k_i) \right\}.$$

From the definition of Bessel functions and the above equality the proposition follows. #

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An efficient algorithm for linear programming

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Abstract. A simple but efficient algorithm is presented for linear programming. The algorithm computes the projection matrix exactly once throughout the computation unlike that of Karmarkar's algorithm where in the projection matrix is computed at each and every iteration. The algorithm is best suitable to be implemented on a parallel architecture. Complexity of the algorithm is being studied.

Keywords. Direction vector; Karmarkar's algorithm; Moore-Penrose inverse; orthogonal projection; projection matrix; projective transformation.

1. Introduction

In 1979 Khachiyan gave an algorithm called ellipsoid algorithm for linear programming [1]. It has polynomial-time complexity. But it is found to be not superior to the simplex algorithm in practice. A new polynomial-time algorithm based on projective transformation technique was published by Karmarkar 1984 [2]. It is being said that this algorithm is superior to the simplex algorithm even in practice.

In this paper an algorithm is presented [3]. We feel that this algorithm is efficient than the existing algorithms because

1. it computes the projection matrix $P = I - A^+ A$, which requires $O(mn^2)$ operations, exactly once throughout the computation and needs only $O(n^2)$ operations per iteration.
2. it makes use of the information that is available from the recently computed point.

Note that Karmarkar's algorithm computes the projection matrix at each iteration and hence requires $O(n^{2.5})$ operations per iteration.

2. Main idea of the algorithm

Consider the following problem

$$\begin{aligned} &\text{minimize} && C'X \\ &\text{subject to} && X \geq 0. \end{aligned} \tag{P1}$$

This problem is trivial because the solution is obtained by setting those components

of $\mathbf{X}$ to zero that correspond to positive components of $\mathbf{C}$ and other components of $\mathbf{X}$ to 'infinity'. We defer from the usual practice to solve the above problem by an iterative method with an initial feasible solution $\mathbf{X}^0 > \mathbf{0}$. The method is described in the following algorithm.

Algorithm A1

Step 1. Compute an initial feasible solution $\mathbf{X}^0 > \mathbf{0}$.

Step 2. Set $\mathbf{Y} = \mathbf{C}$ and $K = 0$.

Step 3. Compute

$$\lambda = \min \left\{ \frac{X_i^k}{Y_i} : Y_i > 0 \right\}$$

Step 4. Compute

$$\mathbf{X}^{k+1} = \mathbf{X}^k - \varepsilon \lambda \mathbf{Y}$$

where $0 < \varepsilon < 1$.

Step 5. If the optimum is achieved then stop.

Step 6. Set $K = K + 1$, $D_k = \text{diag}(X_1^k X_2^k X_3^k \dots X_n^k)$

Step 7. $\mathbf{Y} = D_k \mathbf{C}$, go to step 3.

$D_k \mathbf{C}$ can be thought of angular projection of $\mathbf{C}$ onto the coordinate axis.

Now consider the original problem

$$\text{minimize } \mathbf{C}'\mathbf{X}$$

$$\text{subject to } \mathbf{A}\mathbf{X} = \mathbf{b},$$

$$\mathbf{X} \geq \mathbf{0}. \quad (\text{P2})$$

Let $A^+ = A'(AA')^{-}$, — denotes the generalized inverse. A^+ is called the Moore-Penrose inverse of A . Also, let $P = I - A^+ A$. It can be proved that the columns of P span the null space of A and are normals to the hyperplanes $X_i = 0$ in the null space of A . P is the projection operator that projects every vector in R^n orthogonally onto the null space of A .

To use Algorithm A1 to solve P2, we need to have the initial feasible solution $\mathbf{X}^0$ such that $\mathbf{A}\mathbf{X}^0 = \mathbf{b}$ and the direction vector $\mathbf{Y}$ not in R^n but in the null space of A . This can be achieved by operating P on this vector. With this explanation we now give an algorithm to solve P2.

3. The algorithm

3.1 Algorithm A2

Step 1. Compute an initial feasible solution

$$\mathbf{X}^0 = (X_1^0 X_2^0 X_3^0 \dots X_n^0)$$

such that $X_i^0 > 0$ for $i = 1, 2, \dots, n$.

Step 2. Compute the projection operator

$$P = I - A^+ A$$

Step 3. Compute

$$Cp = PC$$

$$Y = \frac{Cp}{\|Cp\|}$$

Step 4. Set $K = 0$

Step 5. Compute

$$\lambda = \min \left\{ \frac{X_i^k}{Y_i}; Y_i > 0 \right\}$$

the problem has unbounded solution if all $Y_i \leq 0$ and at least one $Y_i < 0$. If all $Y_i = 0$ then X^k is a solution.

Step 6. Compute

$$X^{k+1} = X^k - \varepsilon \lambda Y$$

where $0 < \varepsilon < 1$.

Step 7. If

$$\|X^{k+1} - X^k\| < 2^{-L},$$

where L is defined as in [2], then stop.

Step 8. Set

$$K = K + 1, D_k = \text{diag}(d_{ii}),$$

where $d_{ii} = X_i^k / \sqrt{P_{ii}}$ and is the distance between the point X^k and the hyperplane $X_i = 0$. P_{ii} is positive because $P^2 = P$.

Step 9. Compute

$$Y = \frac{PD_k Cp}{\|PD_k Cp\|}$$

go to step 5.

Distance δ between the point X^k and the hyperplane $X_i = 0$ is calculated as follows. Consider

$$X^k - \delta \frac{P_i}{\|P_i\|} \Rightarrow X_i^k - \delta \sqrt{P_{ii}} = 0$$

Since $P_i / \|P_i\|$ is the unit normal to the hyperplane $X_i = 0$ and $\|P_i\| = \sqrt{P_{ii}}$. This implies

$$\delta = \frac{X_i^k}{\sqrt{P_{ii}}}.$$

Step 3 computes the orthogonal projection of C onto the null space of A and the initial direction vector, step 5 computes the step length, step 6 a new feasible solution at which the objective function value is less than that of the previous value and step 9 the new direction vector and its projection on to the null space of A . It can be easily seen that the computation of the algorithm is dominated by the computation of the new direction vector which requires only $O(n^2)$ operations. Whereas in Karmarkar's algorithm it requires $O(n^{2.5})$ operations.

3.2 Correctness of algorithm A2

Theorem 1. Let P2 have a bounded solution. Then algorithm A2 converges to a solution of P2.

Proof.

(i) *Nonnegativity conditions*

Since $X^0 > 0$,

$$\lambda = \min \left\{ \frac{X_i^k}{Y_i} : Y_i > 0 \right\},$$

and $\varepsilon < 1$ it follows that $X^{k+1} > 0$ for each k and hence

$$\lim_{k \rightarrow \infty} X_i^k \geq 0$$

for each i .

(ii) *Equality constraints*

Since $AX^0 = b$ and for $k=0$

$$AY = \frac{ACp}{\|Cp\|} = \frac{APC}{\|Cp\|} = 0 \quad \text{since } Cp = PC$$

it follows that $AX^1 = b$.

For $k \geq 1$,

$$AY = \frac{1}{\|PD_k Cp\|} APD_k Cp = 0$$

and hence $AX^k = b$.

(iii) *Optimality of the objective function*

For $k=0$, we have

$$\begin{aligned} C'X^k - C'X^{k+1} &= \varepsilon \lambda C' \frac{Cp}{\|Cp\|} \\ &= \frac{1}{\|Cp\|} \varepsilon \lambda C' Cp \\ &= \frac{1}{\|Cp\|} \varepsilon \lambda C' PCp \quad \text{since } PCp = Cp \\ &= \frac{1}{\|Cp\|} \varepsilon \lambda C' P' Cp \quad \text{since } P = P' \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\|\mathbf{Cp}\|} \varepsilon \lambda (PC)^t \mathbf{Cp} \\
&= \frac{1}{\|\mathbf{Cp}\|} \varepsilon \lambda \mathbf{Cp}^t \mathbf{Cp} \text{ since } PC = \mathbf{Cp} \\
&= \varepsilon \lambda \|\mathbf{Cp}\| > 0.
\end{aligned}$$

For $k \geq 1$,

$$\begin{aligned}
\mathbf{C}^t \mathbf{X}^k - \mathbf{C}^t \mathbf{X}^{k+1} &= \varepsilon \lambda \mathbf{C}^t \mathbf{Y} \\
&= \frac{1}{\|PD_k \mathbf{Cp}\|} \varepsilon \lambda \mathbf{C}^t PD_k \mathbf{Cp} \\
&= \frac{\varepsilon \lambda}{\|PD_k \mathbf{Cp}\|} \mathbf{Cp}^t D_k \mathbf{Cp} \\
&= \frac{\varepsilon \lambda}{\|PD_k \mathbf{Cp}\|} (\sqrt{D_k} \mathbf{Cp})^t (\sqrt{D_k} \mathbf{Cp})
\end{aligned}$$

where $\sqrt{D_k} = \text{diag}(\sqrt{d_{ii}})$. $\sqrt{d_{ii}}$ is real because $d_{ii} > 0$. Hence

$$\mathbf{C}^t \mathbf{X}^k - \mathbf{C}^t \mathbf{X}^{k+1} = \frac{\varepsilon \lambda}{\|PD_k \mathbf{Cp}\|} \|\sqrt{D_k} \mathbf{Cp}\|^2 > 0.$$

Therefore $\mathbf{C}^t \mathbf{X}^k$ is a decreasing sequence. Since P2 is assumed to have a bounded solution it follows that $\mathbf{C}^t \mathbf{X}^k$ is bounded below. Hence $\mathbf{C}^t \mathbf{X}^k$ converges.

3.3 Complexity of algorithm A1

Let L be such that 2^L is numerical infinity and 2^{-L} is numerical zero.

Theorem 2. *Algorithm A1 converges to the solution of P1 in $O(L)$ steps.*

Proof. Without loss of generality, we assume $\mathbf{X}^0 = (1 \ 1 \ 1 \dots 1)^t$. Then

$$\begin{aligned}
\mathbf{X}^1 &= \mathbf{X}^0 - \varepsilon \lambda_1 \mathbf{C}, \\
\lambda_1 &= \min \left\{ \frac{1}{C_i} : C_i > 0 \right\}.
\end{aligned}$$

Let $\lambda_1 = 1/C_j$. Then

$$X_i^1 = 1 - \varepsilon \frac{C_i}{C_j}.$$

Consider

$$\begin{aligned}
X_i^2 &= X_i^1 - \varepsilon \lambda_2 X_i^1 C_i, \\
\lambda_2 &= \min \left\{ \frac{X_i^1}{X_i^1 C_i} : C_i > 0 \right\} \text{ since } X_i^1 > 0 \\
&= \frac{1}{C_j}.
\end{aligned}$$

Therefore

$$\begin{aligned} X_i^2 &= X_i^1(1 - \varepsilon \lambda_2 C_i) \\ &= X_i^1 \left(1 - \varepsilon \frac{C_i}{C_j} \right) \\ &= \left(1 - \varepsilon \frac{C_i}{C_j} \right)^2. \end{aligned}$$

In general,

$$X_i^k = \left(1 - \varepsilon \frac{C_i}{C_j} \right)^k$$

If $C_i < 0$ then X_i^k converges to 2^L for

$$k = \frac{L}{\log_2 \left(1 - \varepsilon \frac{C_i}{C_j} \right)}$$

because

$$\begin{aligned} \left(1 - \varepsilon \frac{C_i}{C_j} \right)^k &\geq 2^L \Rightarrow k \log_2 \left(1 - \varepsilon \frac{C_i}{C_j} \right) \geq L \\ &\Rightarrow k \geq \frac{L}{\log_2 \left(1 - \varepsilon \frac{C_i}{C_j} \right)}. \end{aligned}$$

Similarly, if $C_i > 0$ then X_i^k converges to 2^{-L} for

$$k = \frac{-L}{\log_2 \left(1 - \varepsilon \frac{C_i}{C_j} \right)}$$

Therefore, Algorithm A1 converges to the solution in $O(L)$ steps.

4. Conclusions

Complexity of Algorithm A2 is being studied. Observe that $\lambda_1 = \lambda_2 = \dots$ in Algorithm A1. This observation may be useful in establishing the complexity of Algorithm A2. We feel that the path followed by our algorithm is the same as that of Karmarkar's algorithm. Therefore, Algorithm A2 takes at the most $O(n)$ steps to reach the optimum and hence the complexity of Algorithm A2 is $O(n^3)$. In practice the calculation needs to be done in high precision.

Modifying the direction vector $\mathbf{Y} = (PD_k \mathbf{Cp}) / \|PD_k \mathbf{Cp}\|$ in Step 9 of Algorithm A1 as $\mathbf{Y} = (PD_k^* \mathbf{Cp}) / \|PD_k^* \mathbf{Cp}\|$ by introducing a real parameter γ_k , another algorithm will be obtained. The resulting algorithm will be better in performance and can handle practical problems with the existing precision if the optimal values can be computed for γ_k . Further details on computing γ_k will be discussed in a future correspondence.

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Erratum

Verschiebung and Frobenius operators by K Varadarajan *Proc. Indian Acad. Sci. (Math. Sci.)* **100** (1990) pp. 37–43

Lines 4 and 5 on page 39:

The map $E: W(A) \rightarrow 1 + tA[[t]]$ is a ring isomorphism when $1 + tA[[t]]$ is considered as a ring with $\oplus$ as the addition and $*$ defined below as the multiplication.

$$f(t)*g(t) = 1/f(t) \circ g(t),$$

where $\circ$ is the product operation defined in §1.



SUBJECT INDEX

- A general class of polynomials
 - On the inverse Laplace transform of the product of a general class of polynomials and the multi-variable H-function 21
- Affine linearization
 - Affine Poisson structures 189
- Almost periodic functions
 - Compact operators on a Banach space into the space of almost periodic functions 25
- Anderson model
 - Anderson model with decaying randomness existence of extended states 285
- Arithmetic-geometric mean
 - On the ratio of two blocks of consecutive integers 107
- Bessel function
 - Some results involving Bessel function and Fox's H-function of two variables 255
- Borel automorphism
 - On the existence of a finite invariant measure 203
- Borel space
 - On the existence of a finite invariant measure 203
- Central extensions
 - Affine Poisson structures 189
- Combinatorial coefficients
 - Extension of certain types of generating relations 133
- Compact operators
 - Compact operators on a Banach space into the space of almost periodic functions 25
- Compactified Jacobian
 - Singular pencils of quadrics and compactified Jacobians of curves 95
- Compressible flow
 - On shock dynamics 87
- Compressible set
 - On the existence of a finite invariant measure 203
- Continuous selections
 - On infinite-dimensional control system with state and control constraints 65
- Controllability
 - On infinite-dimensional control systems with state and control constraints 65
- Convex sub-differential
 - On infinite-dimensional control systems with state and control constraints 65
- Covariant system
 - Pairs of Π_1 factors 157
- Critical exponent
 - Critical Sobolev exponent problem in $\mathbb{R}_n$ ($N \geq 4$) with Neumann boundary condition 275
- Cross-dual of a sequence space
 - Köthe spaces and topological algebra with bases 259
- Cutlocus
 - A class of totally geodesic foliations of Lie groups 57
- de Rham complex
 - Decomposition of the de Rham complex 103
- Decaying randomness
 - Anderson model with decaying randomness existence of extended states 285
- Deviating arguments
 - Existence of solution for nonlinear Volterra integral equations 179
- Direction vector
 - An efficient algorithm for linear programming 295
- Elementary Diophantine approximations
 - On the ratio of two blocks of consecutive integers 107
- Equicontinuous and absolute basis
 - Köthe spaces and topological algebra with bases 259
- Erratum 303
- Expansion formulae
 - Some results involving Bessel function and Fox's H-function of two variables 255
- Exponential Diophantine equations
 - On the ratio of two blocks of consecutive integers 107
- Extended state
 - Anderson model with decaying randomness existence of extended states 285
- Fixed point algorithm
 - $\tilde{K}_3$ - A new triangulation of R^n 79
- Flatness condition
 - Critical Sobolev exponent problem in $\mathbb{R}_n$ ($N \geq 4$) with Neumann boundary condition 275

Foliations		Karmarkar's algorithm	
A class of totally geodesic foliations of Lie groups	57	An efficient algorithm for linear programming	295
Fourier series		Khinchine's theorem	
Some results involving Bessel function and Fox's		Diophantine approximation by linear forms on	
H-function of two variables	255	manifolds	221
Fourier transform		Köthe sequence spaces	
Some uncertainty inequalities	137	Köthe spaces and topological algebra with bases	259
Riesz means for the sublaplacian on the Heisenberg			
group	147		
Fox's H-function		L^1 origin	
Some results involving Bessel function and Fox's		Integrability and L^1 -convergence of sine series	
H-function of two variables	255	with generalized quasi-convex coefficients	245
G_1 - and G_∞ conditions		Laguerre expansion	
Köthe spaces and topological algebra with bases		Some uncertainty inequalities	137
	259	Multipliers for the Weyl transform and Laguerre	
Gas bubble		expansions	9
Effect of a viscous fluid flow past a spherical gas		Laguerre polynomials	
bubble on the growth of its radius	185	Extension of certain types of generating relations	133
Generating relations		Lattice path enumeration	
Extension of certain types of generating relations	133	Extension of certain types of generating relations	133
Geometric criterion		Lebesgue-Stieltjes integral	
Criterion for smoothness of Schubert varieties in		Lebesgue-Stieltjes integral inequalities in several	
$Sl(n)/B$	45	variables with retardation	231
Grid size		Legendre functions of the second kind	
$\tilde{K}'_3$ - A new triangulation of $\mathbb{R}^n$	79	Relative extrema of Legendre functions of the	
Gronwall inequality		second kind	1
Lebesgue-Stieltjes integral inequalities in several		Lie algebras	
variables with retardation	231	Affine Poisson structures	189
Grothendieck-Pietsch nuclearity criteria		Liftings of varieties	
Köthe spaces and topological algebra with bases		Decomposition of the de Rham complex	103
	259	Linear forms in logarithms	
Group-like object		On the ratio of two blocks of consecutive integers	107
Pairs of II_1 factors	157	Locally convex algebra	
Hausdorff dimension		Köthe spaces and topological algebra with bases	259
Diophantine approximation by linear forms on		Lower semicontinuous multifunction	
manifolds	221	On infinite-dimensional control systems with	
Heisenberg group		state and control constraints	65
Some uncertainty inequalities	137	Manifolds	
Hermite expansion		Diophantine approximation by linear forms on	
Some uncertainty inequalities	137	manifolds	221
Hodge decomposition		Metric diophantine approximation	
Decomposition of the de Rham complex	103	Diophantine approximation by linear forms on	
Inequalities		manifolds	221
Relative extrema of Legendre functions of the		Momentum map	
second kind	1	Affine Poisson structures	189
Invariant measure		Monotonicity	
On the existence of a finite invariant measure	203	Relative extrema of Legendre functions of the	
Involution theorems		second kind	1
Affine Poisson structures	189	Moore-Penrose inverse	
Irreducible subfactor		An efficient algorithm for linear programming	295
Pairs of II_1 factors	157		

Multipliers		Relaxed system	
Multipliers for the Weyl transform and Laguerre expansions	9	On infinite-dimensional control systems with state and control constraints	65
Necklace ring		Representation	
Verschiebung and Frobenius operators	37	Riesz means for the sublaplacian on the Heisenberg group	147
Neumann boundary		Retardation	
Critical Sobolev exponent problem in $\mathbb{R}_n (N \geq 4)$ with Neumann boundary condition	275	Lebesgue-Stieltjes integral inequalities in several variables with retardation	231
Nodal hyperelliptic curve		Riesz means	
Singular pencils of quadrics and compactified Jacobians of curves	95	Riesz means for the sublaplacian on the Heisenberg group	147
Non-linear waves		Ring of Witt vectors	
On shock dynamics	87	Verschiebung and Frobenius operators	37
Nonlinear Volterra integral equation		Schauder basis	
Existence of solution for nonlinear Volterra integral equations	179	Köthe spaces and topological algebra with bases	259
Nuclear locally convex space		Schubert variety	
Köthe spaces and topological algebra with bases	259	Criterion for smoothness of Schubert varieties in $Sl(n)/B$	45
Orthogonal basis		Shock propagation	
Köthe spaces and topological algebra with bases	259	On shock dynamics	87
Orthogonal projection		Singularity link	
An efficient algorithm for linear programming	295	Vector fields and framings on isolated complete intersection singularities	53
Oscillatory singular integrals		Smoothness	
Multipliers for the Weyl transform and Laguerre expansions	9	Criterion for smoothness of Schubert varieties in $Sl(n)/B$	45
Poisson structures		Special functions	
Affine Poisson structures	189	Some results involving Bessel function and Fox's H-function of two variables	255
Projection matrix		Sublaplacian	
An efficient algorithm for linear programming	295	Riesz means for the sublaplacian on the Heisenberg group	147
Projections		Summability	
Riesz means for the sublaplacian on the Heisenberg group	147	Multipliers for the Weyl transform and Laguerre expansions	9
Projective transformation		Symplectic structures	
An efficient algorithm for linear programming	295	Affine Poisson structures	189
Quasi-convex sequence		The Laplace transform	
Integrability and L^1 -convergence of sine series with generalized quasi-convex coefficients	245	On the inverse Laplace transform of the product of a general class of polynomials and the multi-variable H-function	21
Rank-1 Riemannian symmetric space		The multivariable H-function	
A class of totally geodesic foliations of Lie groups	57	On the inverse Laplace transform of the product of a general class of polynomials and the multi-variable H-function	21
Rayleigh-Plesset formula		Triangulation	
Effect of a viscous fluid flow past a spherical gas bubble on the growth of its radius	185	$\tilde{K}'_3$ - A new triangulation of R^n	79
Relaxation		Twisted convolution	
On infinite-dimensional control systems with state and control constraints	65	Multipliers for the Weyl transform and Laguerre expansions	9
		Unconditional convergence	
		Köthe spaces and topological algebra with bases	259

Unital power series ring		Vector field	
Verschiebung and Frobenius operators	37	Vector fields and framings on isolated complete intersection singularities	53
Upper semicontinuous multifunction			
On infinite-dimensional control systems with state and control constraints	65	Weyl transform	
		Multipliers for the Weyl transform and Laguerre expansions	9

AUTHOR INDEX

- | | | | |
|---|-----|---|-----|
| Adimurthi | | Krishna M | |
| Critical Sobolev exponent problem in $\mathbb{R}^n (N \geq 4)$ | | Anderson model with decaying randomness existence of extended states | 285 |
| with Neumann boundary condition | 275 | | |
| Askey Richard | | Lakshmibai V | |
| Relative extrema of Legendre functions of the second kind | 1 | Criterion for smoothness of Schubert varieties in $Sl(n)/b$ | 45 |
| Babu Ram | | Mao Xureong | |
| <i>see</i> Suresh Kumari | 245 | Lebesgue-Stieltjes integral inequalities in several variable with retardation | 231 |
| Bajpai S D | | Nadkarni M G | |
| Some results involving Bessel function and Fox's H-function of two variables | 255 | On the existence of a finite invariant measure | 203 |
| Balachandran K | | | |
| Existence of solution for nonlinear Volterra integral equations | 179 | Papageorgious Nikolaos S | |
| Bawa N P S | | On infinite-dimensional control systems with state and control constraints | 65 |
| <i>see</i> Shukla D P | 79 | Parekh M N | |
| Bhaskara K H | | <i>see</i> Deshpande M V | 25 |
| Affine Poisson structures | 189 | Prasad Phoolan | |
| Bhatt Subhash J | | On shock dynamics | 87 |
| Köthe spaces and topological algebra with bases | 259 | <i>see</i> Srinivasan R | 93 |
| Bhosle Usha N | | Raina R K | |
| Singular pencils of quadrics and compactified Jacobians of curves | 95 | Extension of certain types of generating relations | 133 |
| Chakraborty B B | | Rynne B P | |
| Effect of a viscous fluid flow past a spherical gas bubble on the growth of its radius | 185 | <i>see</i> Dodson M M | 221 |
| Deheri G M | | Sandhya B | |
| <i>see</i> Bhatt Subhash J | 259 | <i>see</i> Lakshmibai V | 45 |
| Deshpande M V | | Santhanam G | |
| Compact operators on a Banach space into the space of almost periodic functions | 25 | A class of totally geodesic foliations of Lie groups | 57 |
| Dodson M M | | Saradha N | |
| Diophantine approximation by linear forms on manifolds | 221 | On the ratio of two blocks of consecutive integers | 107 |
| Gupta K C | | Seade Jose | |
| On the inverse Laplace transform of the product of a general class of polynomials and the multi-variable H-function | 21 | Vector fields and framings on isolated complete intersection singularities | 53 |
| Illamaran S | | Shorey T | |
| <i>see</i> Balachandran K | 179 | <i>see</i> Saradha N | 107 |
| | | Shukla D P | |
| | | $\tilde{K}'_3 - A$ new triangulation of R^n | 79 |

Soni R C		expansions	9
<i>see</i> Gupta K C	21	Some uncertainty inequalities	137
Srinivas V		Riesz means for the sublaplacian on the Heisenberg group	147
Decomposition of the de Rham complex	103		
Srinivasan R		Varadarajan K	
Corrections to some expressions in "On the propagation of a multi-dimensional shock of arbitrary strength"	93	Verschiebung and Frobenius operators	37
Sunder V S		Venkaiiah V Ch	
Pairs of Π_1 -factors	157	An efficient algorithm for linear programming	295
Suresh Kumari		Vickers J A G	
Integrability and L^1 -convergence of sine series with generalized quasi-convex coefficients	245	<i>see</i> Dodson M M	221
Thangavelu S		Yadava S L	
Multipliers for the Weyl transform and Laguerre		<i>see</i> Adimurthi	275

